RECENT DEVELOPMENTS IN NONLINEAR ANALYSIS

Habib Ammari
Abdelmoujib Benkirane
Abdelfettah Touzani
editors

World Scientific

RECENT DEVELOPMENTS IN NONLINEAR ANALYSIS

This page intentionally left blank

RECENT DEVELOPMENTS IN NONLINEAR ANALYSIS

Proceedings of the Conference in Mathematics and Mathematical Physics

Fez, Morocco   28 – 30 October 2008

editors

Habib Ammari
École Polytechnique, France

Abdelmoujib Benkirane
Faculty of Sciences of Fez, Morocco

Abdelfettah Touzani
Faculty of Sciences of Fez, Morocco

World Scientific

NEW JERSEY • LONDON • SINGAPORE • BEIJING • SHANGHAI • HONG KONG • TAIPEI • CHENNAI

Published by

World Scientific Publishing Co. Pte. Ltd.

5 Toh Tuck Link, Singapore 596224

USA office: 27 Warren Street, Suite 401-402, Hackensack, NJ 07601

UK office: 57 Shelton Street, Covent Garden, London WC2H 9HE

British Library Cataloguing-in-Publication Data
A catalogue record for this book is available from the British Library.

RECENT DEVELOPMENTS IN NONLINEAR ANALYSIS
Proceedings of the Conference in Mathematics and Mathematical Physics

ISBN-13 978-981-4295-56-7
ISBN-10 981-4295-56-6

Printed in Singapore.

PREFACE

The *Conference of Mathematics and Mathematical Physics* was held in Fez, Morocco during the period 28–30 October, 2008. It was part of the *5th Congress of the Scientific Research Outlook and Technology Development in the Arab World*.

The present volume contains the texts of a selection from among the many interesting presentations delivered at the conference.

The main objectives of the conference were to promote exchange of the most recent results and emerging ideas and also to merge expertise of researchers in the field not only from the Arab countries but from around the World.

The conference covered in a broad and balanced fashion both the theoretical and applied parts of modern nonlinear analysis.

Both the Local Organizing and Programme Committees deserve great thanks in creating a well-run and very productive conference, with an exciting programme of keynote lectures and contributed talks. It is a pleasure to thank all of them for their hard work. The support of the Arab Science and Technology Foundation and its president Dr. Abdalla A. Alnajjar are also gratefully acknowledged.

August 6th, 2009

H. Ammari — Ecole Polytechnique, France
A. Benkirane and A. Touzani — Faculty of Sciences of Fez, Morocco

This page intentionally left blank

CONTENTS

Preface v

M/v 1
A. Bahri

Nonlinear elliptic equations of infinite order 17
A. Benkirane, M. Chrif, and S. El Manouni

Some remarks on a sign condition for perturbations of nonlinear problems 30
A. Benkirane, J. Benouna, and M. Rhoudaf

On the principle eigencurve of a coupled system 43
A. El Khalil

Parabolic inclusions with nonlocal conditions 54
A. Boucherif

Periodic solutions of nonlinear parabolic equations with measure data and polynomial growth in $|\nabla u|$ 72
A. El Hachimi and A. Lamrani Alaoui

Existence and L^∞-regularity results for some nonlinear elliptic Dirichlet problems 88
A. Youssfi

Analysis of a new mixed formulation of the obstacle problem 101
B. Seddoug, A. Addou, and A. Lidouh

An obstacle problem via a sequence of penalized problems 123
E. Azroul and M. Rhoudaf

Numerical analysis of slopes stability under seismic loading in Lebanon 141
F. Hage Chehade and M. Sadek

Existence and uniqueness result for a class of nonlinear parabolic equations with L^1 data 150
H. Redwane

Existence and uniqueness of solutions of some nonlinear equations in Orlicz spaces and weighted Sobolev spaces 170
L. Aharouch, A. Benkirane, J. Bennouna, and A. Touzani

Existence of solutions for variational degenerated unilateral problems 181
L. Aharouch, E. Azroul, and M. Rhoudaf

Existence and multiplicity results for some $p(x)$-Laplacian Neumann problems 205
M. Bendahmane, M. Chrif, and S. El Manouni

Positive solutions with changing sign energy to nonhomogeneous elliptic problem of fourth order 217
M. Talbi and N. Tsouli

A survey on potential theory on Orlicz spaces 234
N. Aïssaoui

Back on stochastic model for sandpile 266
N. Igbida

On radial solutions for Navier boundary eigenvalue problem with p-biharmonic operator 278
S. El Habib and N. Tsouli

Floquet states of periodically time-dependent harmonic oscillators 290
Y. Achkar, S. Sayouri, and A. L. Marrakchi

Renormalized solutions of nonlinear degenerated parabolic problems: Existence and uniqueness 327
Y. Akdim, J. Bennouna, M. Mekkour, and M. Rhoudaf

Solvability of quasilinear degenerated elliptic equations with L^1 data 344
Y. Akdim

Combined fuzzy sliding mode controller for a class of nonlinear systems 355
Y. Alaoui Hafidi, J. Boumhidi, and I. Boumhidi

Author Index 365

This page intentionally left blank

M/v

A. Bahri

Department of Mathematics
Rutgers University, New Brunswick, NJ 08903, USA
E-mail: abahri@math.rutgers.edu

Given a three dimensional closed contact manifold (M^3, α) and a nowhere singular Morse vector-field v in its kernel, we sketch the construction of the space M/v discussed in [4]. We also introduce spaces of immersed curves in M/v and an action functional on these spaces. This is the first step in the completion of a program aimed at computing the homology for contact forms defined in [2] and [7].

Keywords: M/v; Homology for contact forms.

1. Introduction

We consider in this paper a three dimensional closed manifold M and a contact form α on M. We assume that there is a nowhere zero vector-field v in $ker\alpha$, which we also assume to be Morse-Smale. v might have some hyperbolic orbits around which $ker\alpha$ "does not turn well", see [1], I.11, [2]. We sketch in what follows a method in order to compute the contact homology that we have defined in [2], [3]. As we have indicated in earlier papers [3], [4], this computation requires the use of the space M/v, a highly pathological, non Hausdorff space. We thus have to devote some time to define such a space, or subsets of this space in a manner that suits our purpose.

The idea here has two sides: on one hand, a proper, acceptable definition of the space of orbits mod v, M/v, cannot be given. But a "hybrid" representation of this space, using partly a section to v and partly periodic orbits can be provided.

The next step is then to consider the $\mathcal{Z}$-structure over such a "section" provided by the contact structure; namely, our "section" will be defined in a v-invariant subset of M, where α "turns well" along v. Accordingly, every point x on a v-orbit originating in our "section" will have infinitely many "coincidence points", see [1], Definition 0.1, p. I.7 ,[2], Definition 9, p. 196 (a "coincidence point" of [2] is an "oriented coincidence point" of [1]; the discrepancy is unfortunate, but meaningless); these are points z_k such that $ker\alpha$ has rotated $k\pi$ from x to z_k, $k \in Z$ ($2k\pi$ for [2]. We will use here the original terminology of [1]).

Let ξ be the Reeb vector-field of α. Assume that $\beta = d\alpha(v,.)$ is, in this subset of M defined by the v-orbits originating at this "section", a contact form with the same orientation as α.

We define in what follows path spaces adjusted to this $\mathcal{Z}$ structure. These path spaces are the natural generalization of the space of immersed curves in S^2 of Maslov index zero. This space of immersed curves appears in a natural way [1], [2], when we study the standard contact structure of S^3 and we take for v a vector-field defining a Hopf fibration in its kernel.

Among all the contact forms of this contact structure, there is a special subset which corresponds to the contact forms of this contact structure which are invariant through the antipodal map. These are the "symmetric" contact forms of this contact structure. They enjoy additional symmetries and the study of their Reeb vector-fields and their periodic orbits is greatly simplified as a consequence.

The antipodal map is a special map that generalizes to the most general framework of a contact structure and of a vector-field v, maybe non-singular, of its kernel. Accordingly, the notion of a symmetric α generalizes, as we will see, albeit under some restrictions. An averaging procedure (section 2) allows to define such a notion.

We then sketch the definition in section 3 (the logical order should have been the reverse one) of the space M/v. We essentially show how to define a fundamental domain for an iteration map along v along e.g an attractive orbit of O_1 and we show how we can evolve from there and "travel" using time maps of the one parameter group of v to the hyperbolic orbits and to the repulsive ones, "filling" sections etc.

In the last section, section 4, we sketch the definition of a "symmetric" functional J_s on a space of curves slightly smaller than the space of immersed curves of M/v and we indicate why J_s should become very large or tend to ∞ as we tend to the (hyperbolic to the least, and after some adjustments [6]) traces of the periodic orbits of v in M/v.

2. The path spaces, the nearly symmetric α

Let us assume that the space M/v has been defined as an "orbifold section" to v, possibly with boundary. Typically, we would think of the standard contact structure of S^3, of v as being a Morse-Smale perturbation in its kernel of a vector-field defining a Hopf fibration, with an attractive periodic orbit O_1 and a repulsive one O_2, see [5], Theorem 1, to find such a vector-field v in an almost explicit form. M/v then can be taken to be a disk transverse to O_1, with boundary O_2.

If we then consider an immersed C^1-closed curve $x(t)$ in this "section", we can lift it above this "section" along v into a C^1-curve $y(t)$ so that $\dot{y}(t)$ reads as $a\xi + bv$, a positive and $y(t)$ is derived from $x(t)$ by v-transport. $y(t)$ is not unique, neither is it necessarily a closed curve. Rather, given one of the lifts $y(t)$, all other lifts are indexed by an integer $k \in Z$ and derived using the map along the v-orbit which assigns to a point x_0 the point x_k uniquely defined by requiring that β (thus ξ) has completed k half-revolutions between x_0 and x_k.

In order to have $y(t)$ closed, we may have to ask that $x(t)$ be a path, rather than a closed curve, starting at a point x_0 and ending at a point x_1, both in the "section" and both on the same v-orbit.

Let T be the map, defined on the "interior of the section", which assigns to x_0 of this "section" the next point x_0' on the v-orbit through x_0 which belongs again to this "section".

If the end point x_1 of the curve $x(t)$ defined above is equal to $T^m(x_0)$ for suitable values of m and if $\dot{x}(1)$, the tangent vector to $x(t)$ at x_1, is equal to $DT^m(\dot{x}(0))$, then the curves $y(t)$ will be closed curves as we will see.

Not only the curve $x(t)$, $t \in [0,1]$, lifts into $y(t)$ and the corresponding family of closed curves above $x(t)$. The curves in this "section" defined by $T^i(x(t))$, $t \in [0,1]$, $i \in \mathcal{Z}$ all lift into the same family of curves $y(t)$. So that we find it natural to introduce:

Definition 2.1. the space of curves $\Lambda_{T^m}(M/v)$ (M/v is our "section") defined as the set of C^1-curves in M/v running from a point x_0 of M/v to the point $T^m(x_0)$.

The map T defines a transformation T_* of this space and we will denote $\Lambda^*_{T^m}(M/v)$ the quotient of this space by T_*.

Our space could be in fact more specific because M/v is typically a stratified space of dimension 2, with boundary one of the attractive or repulsive orbits. Typically, M/v is a disk, with boundary an attractive periodic orbit O_1 for example.

We can arrange so that T^m, restricted to O_1, is the identity map and that ξ rotates exactly $m\pi$ in the v-transport along O_1. It is then natural to consider the space $(M/v)/O_1$, the topological quotient of M/v by its subset O_1 and therefore to introduce the space $\Lambda^*_{T^m}((M/v)/O_1) = \Lambda^*$ of C^1-curves running in $(M/v)/O_1$ from an initial point x_0 to $T^m(x_0)$ mod out by the map T^* (defined as above, but acting on these new spaces).

Embedded into Λ^*, we find the space of C^1-immersed curves Imm^*.

The group S^1 acts on these spaces by time translation. Furthermore, above any given curve in Imm^*, we find a family of closed curves $y(t)$, $t \in [0,1]$, with $\dot{y} = a\xi + bv$, only that the constant a might change with the curve y in the family. a does not change if the form α is "symmetric", that is if, whenever the v-transport, along a v-orbit from x_0 to x_1, maps ξ into $\lambda\xi$, then $\lambda = 1$.

Of course, given a contact structure, a vector-field v in its kernel and a contact form α in this contact structure, we cannot expect α to be "symmetric"; neither can we assume, in general, the existence of a "symmetric" α.

The nearly symmetric α

However, after averaging α, we can assume that α is nearly "symmetric". This averaging procedure goes as follows: considering a point z_0 above M/v, we introduce the points z_i, $i \in [-N, N]$, N large. z_i is defined by the condition that it is the i^{th}-point on the v-orbit such that α is mapped onto $\lambda_i\alpha$ from z_0 to z_i. A candidate in order to replace α at z_0 is $\dfrac{1}{\left(\sum_{i=-N}^{N}\frac{1}{\lambda_i}\right)}\alpha$; this is formally an almost symmetric contact form in the same contact structure than α. It is, by Lemma 1 of [6] v-convex (that is, denoting θ this form that has v in its kernel, $d\theta(v,.)$ is also a contact form with the same orientation than θ) since each of the forms $\lambda_i\alpha$ is v-convex (they are pull-backs of α through v-transport maps); but it has the disadvantage to tend to zero as N tends to ∞ on any v-orbit that is asymptotic to an attractive or a repulsive periodic orbit of v.

The contact form $\left(\displaystyle\sum_{i=-N}^{N}\lambda_i\right)\alpha$ is also nearly symmetric and does converge on any v-orbit tending at ∞ to attractive or repulsive periodic orbits. It is, however, not necessarily v-convex.

Near an attractive or a repulsive periodic orbit of v, a model for (α, v) has been provided in [6] p. 47. This model can be slightly modified so that

$\lambda_i = \bar{\gamma}^i$, with $0 < \bar{\gamma} < 1$. It follows from this model that $\left(\sum_{i=-N}^{N} \lambda_i\right)\alpha$ is also v-convex (terminology of [5], Lemma 1) near the attractive or repulsive periodic orbits of v (taking (α, v) according to the model).

This "nearly symmetric" form also "turns well" along v wherever $ker\alpha$ "turns well" along v. We can then use, as in [6], the second order differential equation along v:

$$(1)[v,[v,\xi]] = -\xi + \gamma(s)[\xi, v] - \gamma'(s)ds(\xi)v$$

with $s(.)$ denoting the length along v on a given piece of v-orbit with a given origin; this differential equation allows to modify α, ξ, see Lemma 1 of [6], by rescaling the rotation of $ker\alpha$ along v.

Starting from the data that we have near each attractive or repulsive periodic orbit, we can evolve along v-orbits and induce a uniform rotation, thereby deriving a nearly symmetric α outside of small tori around the attractive and repulsive periodic orbits of v, along subsets of M where sections to v can be defined; this nearly symmetric α_s is now v-convex, this is embedded in the rescaling with the use of (1); it is furthermore equal to $\left(\sum_{i=-N}^{N} \lambda_i\right)\alpha$ near the repulsive or attractive periodic orbits of v.

If there are hyperbolic orbits of v, global sections outside of the attractive and repulsive periodic orbits might not be available. We need therefore to remove, in a first step of in our construction, the hyperbolic orbits and their stable and unstable manifolds. The rescaling and the definition of an almost symmetric α_s can be completed on the remaining set. In order to extend the definition of our form to the hyperbolic orbits and their stable and unstable manifolds, we follow the construction of [6]; in [6], this construction was carried near a hyperbolic orbit such that $ker\alpha$ did not "turn well" along it. The general idea would be to extend it to all hyperbolic orbits. This construction needs to be carried out in great detail in the present framework (with the aim of deriving a nearly symmetric form in the vicinity of these orbits, or having the associated functional, see section 4 below, tend to ∞ as the curves come to intersect one of these hyperbolic orbits).

Similarly, although the definition of this nearly symmetric contact form is very precise near the attractive and repulsive orbits, the effect of the rescaling, completed above with the use of (1), on the associated variational problem (to be "defined" in section 4, below) and the behavior of its critical points at infinity near the repulsive or attractive periodic orbits of v need to be thoroughly understood.

3. M/v

Let us now enter into more details in the construction of "M/v". We assume that v has two periodic orbits, O_1 which is a attractive and O_2 which is repulsive, and a number of periodic orbits which are hyperbolic; but our v is Morse-Smale, with no cycles. For simplicity, let us assume that we have only two hyperbolic periodic orbits O_3 and O_4, with O_3 dominating O_4, that is the unstable manifold of O_3 intersects the stable manifold of O_4 and not vice-versa. We want to define the hybrid object M/v.

For this, we consider the trace of the stable manifold of O_3 on the boundary ∂T_1 of a torus T_1 transverse to v around O_1.

If the eigenvalues of the Poincaré-return map at O_3 are negative, then the trace of $W_s(O_3)$ on ∂T_1 has only one connected component.

If the Poincaré-return map at O_3 has positive eigenvalues, then there are exactly two connected components to this trace because the stable manifold of O_3, when deprived of O_3, has two connected components and they do not intersect. Each of these is an embedded differentiable closed curve. By standard arguments, it follows that either both curves are embedded isotopic closed curves which both read homotopically as $ma + nb$ on the two S^1-generators of the fundamental group of ∂T_1; or one or both of them are contractible to a point in ∂T_1.

Let us assume that we are in this second case: the results which we derive then can be adapted to the first case.

Let us think of the intersection of the stable manifold of O_4 with ∂T_1. This intersection, though an embedded differentiable curve, is neither closed nor compact. It could also not be connected. We claim that it is made of a finite number of connected components, each of them being an embedded differentiable closed curve whose closure is obtained by addition of one or both connected components of $W_s(O_3) \cap \partial T_1$. This is a fine point which we need to understand.

Let us also assume for simplicity that none of the components of the trace of $W_s(O_3)$ on ∂T_1 is homotopic to zero in ∂T_1.

Assuming in the sequel that both components of $W_s(O_3) \cap \partial T_1$ are not homotopic to zero, they both read $ma + nb$, m, n prime to each other (they are then homotopic since they do not intersect). The components of $W_s(O_4) \cap \partial T_1$ then "spiral" towards these two isotopic embedded curves.

3.1. $\hat{c}$ *and the fundamental domain*

We consider a section $\hat{c}$ to $W_s(O_3)\cap\partial T_1$ in ∂T_1. This section is made of two small embedded pieces of curve defined on two intervals, which are transversal to each of the components of $W_s(O_3)\cap\partial T_1$ and which are connected by two other embedded pieces of curves in $\partial T_1\backslash(W_s(O_3)\cup W_s(O_4))\cap\partial T_1$. We find a closed differentiable embedded curve transverse to both traces of $W_s(O_3)$ and $W_s(O_4)$ on ∂T_1.

We now consider the Poincaré-return map f of v from a section σ to v near O_1 containing $\hat{c}$. σ needs not be transverse to v at O_1, but it should be everywhere else. The choice is very clear if, denoting b the generator transverse to O_1, m in the couple (m,n) defining the homotopy class of $W_s(O_3)\cap\partial T_1$ is non-zero. We can take for σ a disk transverse to O_1 in T_1.

We assume, without loss of generality, that $f(\hat{c})$ is in σ. $f(\hat{c})$ is drawn on the boundary of the solid torus $f(T_1)$. f is generated by the one parameter group of v, γ_s and we thus can write $f=\gamma_{s(.)}$, where $s(.)$ is an appropriate function. We can consider the family of tori $\gamma_{ts(.)}(T_1)$, $t\in[0,1]$. They define a family of curves in σ which define a fundamental domain Δ. We iterate this fundamental domain Δ under positive and negative powers of f. The negative iterations end at O_1. The positive iterates go where they should go, but we are going to track a few portions of Δ under positive iterations.

Observe that $\hat{c}$ intersects each of the components of $W_s(O_3)\cap\partial T_1$ at exactly one point. This point, under positive iterations, will get closer and closer to O_3. Adjusting f nearby O_3 to become the Poincaré-return map of O_3 at one of its points z, in an appropriate section σ_1, a portion of Δ defined by two small transversals in $\hat{c}$, $f(\hat{c})$ containing the points of $W_s(O_3)$ in these sets (there are two of them in each of $\hat{c}$, $f(\hat{c})$) and two other "vertical" pieces of curves connecting these two couples of points(we thereby find a small "rectangle" in σ) will reach σ_1 under iteration from **both** "sides". Indeed, the "vertical"curves connecting the points of $W_s(O_3)$ under (adjusted) iteration will reach O_3 on two distinct sides, thus z in σ_1 from two distinct sides. Just as in [8], in Morse Theory, when considering a non-degenerate critical point, these two "vertical" transversals then "spread" under iteration along $W_u(O_3)\cap\sigma_1$ and its iterates. we will denote this set $\tilde{W}_u(O_3)_z$. It is clear that we have to add it to $\cup f^n(\Delta)$ in order to define M/v.

We now have to evolve to O_4 from O_1 and from O_3. We may assume that the two small transversals in $\hat{c}$ to $W_s(O_3)$ contain all of $W_s(O_4)\cap\hat{c}$ and thus that $\hat{c}$ outside of these small transversals "spouses" $W_s(O_4)\cap T_1$ without intersecting it. Thus, the iterates under f of $W_s(O_4)\cap\hat{c}$ all go to

σ_1 and, from there, should go to O_4.

This situation, the lower stage of the tower of domination, offers a new background which we want to discuss now: the first case is when O_3 dominates O_4 and does not dominate any other hyperbolic orbit. We first elaborate more on this specific situation. We then consider the case when O_3 can dominate more than one, typically two-the arguments then generalize-hyperbolic periodic orbits.

Let us discuss the first case. Let T be the Poincaré-return map of O_4, defined on a section σ_4 to O_4, at a point of O_4. T is generated by the one-parameter group of v and therefore T is homotopic to the identity map, in the set of invertible two dimensional maps. Thus, the differential of T at the origin has a positive determinant. O_4 is hyperbolic, thus the differential of T has two real eigenvalues, of the same sign, one larger than one in absolute value, the other one less than one in absolute value also. The differential of T^2 has only positive eigenvalues. Let us consider $W_s(O_3) \cap \sigma_1$, which is made of one interval, two intervals I^+, I^-, after removing the fixed point. These two intervals are part of two half-lines L^+, L^- which span through the use of the one parameter group of v all of $W_u(O_3)$ (after the addition of O_3). We can imagine that σ_1 has been extended via the use of the Poincaré-return map of O_3 and then the time $1;t$-map of v so that it reaches near O_4 and "touches" σ_2 (σ_1, after iterations, and σ_2 have to intersect since O_3 dominates O_4).

3.2. *How O_1 dominates O_3: A special case*

Let us assume that σ_1 **and** σ_2 can be built as small hyperbolic neighborhoods of pieces of sections to v in $W_u(O_3)$ and $W_s(O_4)$ respectively. I^+, I^- and L^+, L^- have been defined for $W_u(O_3)$, but they can also be defined for O_4. We denote them J^+, J^-, P^+, P^-. We use $I^\pm$, $J^\pm$, $L^\pm$, $P^\pm$ to define these sections. The I's and J's are used when we are on the "sides" of $W_u(O_3)$ and $W_s(O_4)$ that intersect, taking I and J until a common point x which we may choose to be in σ_2, near O_4.

Observe that $W_u(O_3)$ and $W_s(O_4)$ intersect in fact at infinitely many points, a subset of which is derived from x through the use of T.

We then claim-and this claim is more important for the complete understanding of this specific configuration rather than for the definition of M/v-:

Proposition 3.1. *Assume that σ_1 **and** σ_2 can be built as small hyperbolic neighborhoods of pieces of sections to v in $W_u(O_3)$ and $W_s(O_4)$. Then, there* *is another set of points of intersection generated by a y which is not derived from x by iterations. In fact, the intersection points can be viewed as couples of such points (x, y) together with their iterates.*

Proof. Indeed, considering T^2, we know that its differential has positive eigenvalues at zero. Thus T^2 maps J^+ into J^+ and J^- into J^- respectively. Without loss of generality, we may assume that it is J^+ and I^+ that intersect at x. We consider then $W_u(O_3) \cap \sigma_2$ and more specifically the subset corresponding to I^+.

Part of $\hat{c}$ was made of two small pieces of curves transverse to $W_s(O_3) \cap \partial T_1$. we also considered $f(\hat{c})$, thus the image under f of these two small pieces of curves and we connected the ends of each corresponding pair of intervals by "vertical" lines; this yields two pairs (V_1, V_2) and (V_3, V_4) of "vertical lines" which we iterate using f. The related lines are again denoted V_i. Each V_i is as close as we please to the "history", "under iteration", of one of the two points of $\hat{c} \cap (W_s(O_3) \cap \partial T_1)$. This "history" defines a set of lines in $W_s(O_3) \cup W_u(O_3) \cup W_s(O_4) \cup W_u(O_4)$, in fact in the intersection of these sets with the respective sections σ, σ_j. each V_i neither intersects $W_s(O_3)$, nor $W_s(O_4)$.

The smaller fundamental domain defined by the two small "vertical" lines connecting the two intervals and their images under f spread under iteration and "fill" σ_1 (up to the addition of $W_u(O_3) \cap \sigma_1$) and, from there, they "spread" until they "touch" σ_2 just as σ_1 did. From there, we use T and we move to σ_2, which we can view to be bounded on each side by portions of the V_is. Under T^2, the I^+-portion of $W_u(O_3) \cap \sigma_1$ maps into a half-line. Indeed, the image curve does not intersect the V_is, because, if it did, then some points of V_i would not,under reverse iteration, go to O_1, but would go to O_3. It is therefore entirely contained into σ_2. If the differential of T at the origin has positive eigenvalues, then we can use T in lieu of T^2.

This half-"line" intersects J^+, that is the portion of $W_s(O_4) \cap \sigma_2$, into at least one point, namely x,hence also at its iterates under T^2. Under iteration, it "spreads" and its tangent direction becomes pore and more parallel to $W_s(O_4) \cap \sigma_2$.

If the differential of T at zero has negative eigenvalues, then $T^{2k+1}(x)$ is in J^- rather than J^+ and all the points of intersection of $J^+ \cap T^2(I^+)$ would then read as $T^{2k}(x)$ if x and only x spans this intersection. However, the orientation of $T^2(I^+)$ alternates at consecutive intersection points, going from left to right (according to a certain orientation of $W_s(O_4) \cap \sigma_2$) at a point and from right to left at the next point. These consecutive points, because x spans the intersection set, are iterates of each other under T^2.

This yields a contradiction because the eigenvalues of the differential of T^2 at zero are both positive.

The same argument works with T in lieu of T^2 if the differential of T at zero has positive eigenvalues.◁

We now have tracked our fundamental domain under evolution.we have understood how one of the two sides of each of σ_1 and σ_2 are "filled" by the smaller fundamental domain under iteration. The other sides are "filled" because $\hat{c}$ intersects as well the trace of the stable manifolds of O_3, O_4 on ∂T_1 and we can "drive" M/v, with a proper choice of f to bring our set under iteration to "fill" the other sides. We add to these iterated sets the two curves $W_u(O_3) \cap \sigma_1$, $W_u(O_4) \cap \sigma_2$. The construction of M/v is nearly completed in this easier framework, when O_3 dominates only O_4. We still need to understand how this set behaves near O_1, O_2.

3.3. *The general case*

There is a more complicated case, when O_3 dominates more than one hyperbolic orbit; e.g O_3 dominates O_4, O_5, both hyperbolic orbits. The construction of M/v is then greatly simplified by the following Proposition:

Proposition 3.2. *Assume N is a two-dimensional surface transverse to v and intersecting e.g. $W_u(O_3)$ at a point z which is e.g. on the v-orbit of a point of e.g. L^+. Then $N \cap W_u(O_3)$ contains a whole half-line which is an image of L^+ through the one parameter group of v. This statement holds when N is a surface with boundary; then L^+ is replaced by an interval I^+.*

Proof. The intersection of $W_u(O_3)$ and N is a transverse intersection, which therefore yields a differentiable manifold of dimension 1 transverse to v. Let us consider the connected component of this intersection containing z. It is a one dimensional manifold that has a natural projection π over L^+/J^+. π defines a fibration because any two points of $\pi^{-1}(\ell)$, ℓ given in L^+, can never coalesce: v is transverse to this manifold. It follows that this fibration extends throughout L^+, unless it is limited by the boundary of N.◁

3.4. *Outline of the construction of M/v*

The construction of M/v is derived from the choice of the curve $\hat{c}$ on ∂T_1 and from Proposition 3.2. A careful choice of $\hat{c}$ allows us to move from the attractive orbit O_1 to the other hyperbolic orbits and to the repulsive orbit (there could be more attractive and repulsive orbits; we are only describing a simple case here). As we have seen above, near a hyperbolic orbit O_3 which is "directly" dominated by O_1 (i.e there is no intermediate hyperbolic orbit), the iterates of a fundamental domain built using $\hat{c}$ and its image through the Poincaré return map of O_1 will fill a suitable section of this hyperbolic orbit from "the two sides" (the process is different if the Poincaré return map at O_3 has positive or negative determinant; in the first case, the trace of the stable manifold of O_3 on ∂O_4 is connected while, in the second case, it has two connected components).

The construction of M/v is therefore very clear near O_1 and from there to all such O_3s.

Then, from a hyperbolic orbit O_3, we may move to another hyperbolic orbit O_4. There, we use Proposition 3.2 which tells us that, because our iterations of our fundamental domain include a point of the stable manifold of O_4, they will contain all the trace of this stable manifold in an appropriate section. In this way, a full "side" of this section of O_4 will be "filled". The other side will be "filled" either through a similar process, that is starting from O_3; or directly from $\hat{c}$ because $\hat{c}$ is chosen appropriately to intersect the trace of the part of the stable manifold of O_4 that goes directly to ∂O_1.

The process continues in this way (we add to the sections which we encounter the traces of the hyperbolic orbits O_is in these sections), until we have exhausted all hyperbolic orbits. We are then left with O_2. Our fundamental domain under iterations will come to O_2 in a complicated manner, depending also on the linking number of O_1 and O_2, of O_2 with the other O_is.

This is the part of the construction of M/v that requires further study, until we know precisely what is involved in this construction and this object becomes thereby a straightforward object to use.

Despite the fact that our construction of this object is only sketched, we are going to introduce a function on the space of curves Imm^* defined in section 1 on this space and study its properties. This should lead us to a method for the computation of our homology [2], [3].

4. The functional

The natural functional to use on the space of curves $\Lambda_{T^m}(M/v)$ defined in section 2 is the action functional $J(x) = \int_0^1 \alpha_x(\dot{x})$. This functional is invariant under T if the contact form α is "symmetric". However, the contact form built through the averaging procedure of section 2 is only nearly symmetric; it is not symmetric; and the space M/v of section 3 is very well defined only outside of the periodic orbits of v: for example, if we consider the case of the standard contact structure of S^3 and if v is a Morse-Smale perturbation of a vector-field defining a Hopf fibration in $ker\alpha$, O_2, that is the repulsive periodic orbit, is a "boundary" for M/v.

Let us consider in more details the case of the standard contact structure on S^3, with v having two periodic orbits, one attractive O_1 and the other one repulsive O_2. The fundamental observation in this easier framework is that the "symmetrized" α at points x close to O_1 and O_2 (how close depends also on N, the number of iterations of T involved in the "symmetrization" process) reads $\alpha_x = \lambda(x)\alpha_{0x}$, $\lambda(x)$ tending to ∞ as x tends to $O_1 \cup O_2$; that is the "symmetrized" α has a coefficient tending to infinity on the standard contact **form** of S^3.

4.1. *Proposition 4.1*

Some more is true; namely:

Proposition 4.1. *v around O_1 and O_2 may be arranged so that*

i/ *the contact vector field $\xi(x)$ of the "symmetrized" α tends to zero in norm as x tends to $O_1 \cup O_2$.*
In addition,

ii/ *denoting ψ be the map which assigns to a point x the next coincidence point ([1], [2]) on the positive v-orbit through x,*

then, after perturbation, the orbits of ξ do not connect x_0 and $\psi^j(x_0)$ for x_0 in $O_1 \cup O_2$ and for $j \in \mathbf{Z}$.

We give below the proof of Proposition 4.1. Using the results of [6], properly generalized, we expect Proposition 4.1 to hold for every nowhere zero Morse-Smale v in $ker\alpha$. The results of [6] are about the behavior of α around hyperbolic periodic orbits of v along which $ker\alpha$ does not turn well. They of course also apply to hyperbolic orbits around which $ker\alpha$ turns well; that is, it should also be possible in such a case to build "mountains" around these hyperbolic orbits (essentially, i. of Proposition 4.1 should hold around hyperbolic orbits). But, we can then also hope that such "mountains" can be built around the regions where $ker\alpha$ does not turn well along v. Symmetrizing α outside of small neighborhoods of such regions, we would try to perturb such a symmetric α so that ii) of Proposition 4.1 would hold.

Since there is some hope that Proposition 4.1 generalizes, it is useful to prove that this proposition holds in the simpler case of the standard contact structure of S^3 and to indicate, pending the complete and rigorous proof of all details, how the computation of the homology, or the existence of periodic orbits for ξ can be derived from this procedure.

Proof. Given an integer N, as x moves closer to the e.g attractive orbit O_1 of v, the negative iterates of ψ are expanding maps. Assume that $ker\alpha$ rotates twice (v is a perturbation of one of the Hopf-fibrations vector-fields in the kernel of the standard contact form of S^3) along O_1, with a uniform rotation and a uniform coefficient of contraction $-1 < \gamma < 0$ after a quarter of a turn along O_1, so that the coefficient of contraction after a full turn is γ^4. This can be achieved after a suitable perturbation of $v, ker\alpha$ near O_1. The negative iterates of α_0, at a point x_0 of O_1 therefore read as multiples $\gamma^{-i}\alpha_0$. Their sum at the order N (from 0 to $-N$) therefore reads as $\frac{1-\gamma^{-N-1}}{1-\gamma^{-1}}\alpha_0$. The coefficient in front of α_0 tends clearly to ∞ and this fact cannot be destroyed by the contribution of the positive iterates, since ψ is contracting near O_1.

This is the basic phenomenon from which, after some additional work estimating the derivatives along ξ, $[\xi, v]$ of λ (the "symmetrized" α reads $\lambda\alpha_0$), i)follows.

For ii), we observe that, given e.g y a ξ_1-piece of orbit of a symmetric (over a limit process, outside of $O_1 \cup O_2$) contact form α_1 connecting two points x_0 and $\psi^j(x_0)$ of e.g O_1, if we perturb this symmetric α_1 in the vicinity of a point of this ξ_1-piece of orbit into another, symmetric (the symmetry is along ψ and its iterates) form α_2, there will still be, if the intersection problem satisfies the appropriate transversality conditions, in the vicinity of y a ξ_2-piece of orbit connecting two points x_1 and x_2 of O_1, one close to x_0, the other one to $\psi^j(x_0)$; but, generically, x_2 will not be $\psi^j(x_1)$. ii) follows.$\triangleleft$

4.2. *M^*/v and the symmetric α_s*

We now consider the case of a more general nowhere zero vector-field v in the kernel of α, which we assume to be Morse-Smale, having a number of periodic orbits $\cup O_i$, some elliptic, the other ones hyperbolic. Using the propositions and the results of the previous section, we define the space M/v. We can also define the space:

$$M^*/v = M/v \backslash \cup O_i.$$

The averaging procedure for α can be completed on M^*/v.

If we start from a point of $M \backslash \cup O_i$, the positive and negative iterates under ψ^2 (the transport map along v mapping a point x_0 to the next **oriented** coincidence point (see [1], [2]) on the v-orbit through x_0) of a given point end up near the attractive and repulsive orbits of v. It follows that "averaged" limit forms of α, α_s are well defined point-wise on M^*/v, but might have, as one can easily see, discontinuity points along the stable and unstable manifolds of the hyperbolic periodic orbits of v. If there are no such hyperbolic orbits, as in the case of the standard contact structure of S^3, with v a small perturbation of a vector-field in $ker\alpha$ defining a Hopf fibration, then a symmetric form α_s is well-defined and continuous, differentiable on $M \backslash \cup O_i$.

This contact form α_s can be used to define a symmetric functional J_s, that is a symmetrized version of J (see section 1) on the curves of Imm^* which do not intersect $\cup O_i$.

We need therefore to understand the behavior of this functional on the curves of this set that are in the immediate vicinity of curves intersecting $\cup O_i$. Typically, we would want that such curves are "far" from being critical points of J_s and we would in fact want more: namely, we would want that the functional J_s tends to ∞ as we approach such curves.

4.3. *J_s near the hyperbolic orbits*

For hyperbolic orbits (and this also should solve the discontinuity issues involved by the hyperbolic orbits in the definition of α_s), we have devised a construction in [6]. This construction was carried around hyperbolic orbits having the property that $ker\alpha$ does not rotate well along them. Using large amounts of rotation near the attractive and repulsive orbits, one could build a contact form in the same contact structure such that its associated contact vector-field became tiny near these orbits. This construction can be carried out around the other hyperbolic orbits as well, that is around the orbits along which $ker\alpha$ turns well. It should imply that a functional J can be built on Imm^*, extending J_s. J should be very large or ∞ on the curves of Imm^* entering and exiting a small neighborhood of a hyperbolic orbit.

4.4. *Three additional observations*

There are three additional observations that are useful:

First, this procedure should work around the regions of M where $ker\alpha$ does not rotate well along v. The hope is that a construction similar to the one introduced in [6] for the corresponding hyperbolic orbits can be extended to this framework.

Second, i) of Proposition 4.1 above implies that the functional J should be very large or ∞ on a curve of Imm^* that enters a given neighborhood of an attractive or repulsive periodic orbit of v, then intersects this periodic orbit, then exits this given neighborhood (that is J_s should tend to ∞ as we approach such a curve).

Third, ii) of Proposition 4.1 should also generalize into the statement that there are no curve made of pieces of orbits of the symmetric ξ_s up to v-jumps between points x and $\psi^2(x)$ intersecting at least one O_i (repulsive, attractive, or hyperbolic). This is a weaker result than the results foreseen above which say that J is very large or ∞ at curves crossing $\cup O_i$. But it should be a useful additional result.

This provides a very rudimentary version of a scheme in order to compute the homology defined in [3], [4], [7]. But, to the least, one can see here a program; and a glimmer of a reasonable hope that the non-compactness issues can be overcome in Contact Form Geometry.

References

1. R. Adams, *Sobolev spaces*, AC, Press, New York, (1975)
2. L. Aharouch, E. Azroul and M. Rhoudaf *Strongly nonlinear variational parabolic problems in Weighted Sobolev spaces* to appear in AJMAA.
3. Y. Akdim, E. Azroul and A. Benkirane, *Existence Results for Quasilinear Degenerated Equations Via Strong Convergence of Truncations*, Revista Matematica Complutense 17, , N.2, (2004) pp 359-379.
4. Y. Akdim, E. Azroul and A. Benkirane, *Existence of Solution for quasilinear degenerated Elliptic Unilateral Problems*, Ann. Math. Blaise pascal vol 10 (2003) pp 1-20.
5. Y. Akdim, E. Azroul and A. Benkirane, *Existence of solution for quasilinear degenerated elliptic equation*, Electronic J. Diff. Equ. Vol 2001 , N71, (2001) pp 1-19.
6. H. Brezis, and F.E. Browder, *Strongly nonlinear parabolic initial-boundary value problems* , Proc. Nat Acad. Sci. U. S. A. 76 (1976). pp. 38-40.
7. J. Berkovits, V. Mustonen, *Topological degree for perturbation of linear maximal monotone mappings and applications to a class of parabolic problems*,Rend.Mat.Roma,Ser,VII, 12 (1992), pp. 597-621.
8. L. Boccardo, F. Murat, *Strongly nonlinear Cauchy problems with gradient dependt lower order nonlinearity*, Pitman Research Notes in Mathematics, 208 (1988), pp. 347-364.
9. L. Boccardo, F. Murat, *Almost everywhere convergence of the gradients of solutions to elliptic and parabolic equations,* Nonlinear analysis, T.M.A., 19 (1992), n 6, pp. 581-597.
10. A. Dallaglio A. Orsina , *Non linear parabolic equations with natural growth condition and L^1 data.* Nolinear Anal., T.M.A., 27 n1 (1996). pp. 59-73.

11. P. Drabek, A. Kufner and L. Mustonen, *Pseudo-monotonicity and degenerated or singular elliptic operators*, Bull. Austral. Math. Soc. Vol. 58 (1998), 213-221.
12. P. Drabek, A. Kufner and F. Nicolosi, *Non linear elliptic equations, singular and degenerated cases*, University of West Bohemia, (1996).
13. A. Kufner, Weighted Sobolev Spaces, John Wiley and Sons, (1985).
14. R. Landes, *On the existence of weak solutions for quasilinear parabolic initial-boundary value problems,* Proc. Roy. Soc. Edinburgh sect. A. 89 (1981), 217-137.
15. R. Landes, V. Mustonen, *A strongly nonlinear parabolic initial-boundary value problems*, Ark. f. Math. 25. (1987).
16. R. Landes, V. Mustonen, *On parabolic initial-boundary value problems with critical growth for the gradient*, Ann. Inst. H. Poincaré11 (2) (1994) 135-158.
17. J. Leray, J.L. Lions, *Quelques resultats de Viši̇k sur les problèmes elliptiques nonlinéaires par les méthodes de Minty-Browder*, Bull. Soc. Math. France 93 (1995), 97-107.
18. J.L. Lions, *quelques methodes de résolution des problèmes aux limites non linéaires*, Dunod et Gauthiers-Villars, 1969.
19. A. Porretta *Existence results for nonlinear parabilc equations via strong convergence of truncations*, Ann. Mat. Pura. Appl. (1999), pp. 143-172.
20. J. M. Rakotoson *A Compactness lemma for quasilinear problems: application to parabolic equations* J. Funct. Anal. 106 (1992), pp. 358-374.
21. J. Simon *Compact sets in the space* $L^p(0,T,B)$, Ann. Mat. Pura. Appl. 146 (1987), pp. 65-96.
22. M. Rhoudaf *Existence results for Strongly nonlinear degenerated parabolic equations via strong convergence of truncations with* L^1*-data* thèse de Doctorat de M.Rhoudaf, Univ. Sidi Mohamed Ben Abdella. Fès, Maroc 2006.
23. E. Zeidler, *nonlinear functional analysis and its applications, II A and II B, Springer-Verlag (New York-Heidlberg, 1990).*

Nonlinear elliptic equations of infinite order

A. Benkirane

Department of Mathematics, Faculty of Sciences, Atlas-Fes
Fes, 3000, Morocco
E-mail: abd.benkirane@gmail.com

M. Chrif

Department of Mathematics, Faculty of Sciences, Atlas-Fes
Fes, 3000, Morocco
E-mail: moussachrif@yahoo.fr

S. El Manouni

Department of Mathematics, Faculty of Sciences, Al-Imam University
P. O. Box 90950, Riyadh, 11623, Saudi Arabia
E-mail: samanouni@imamu.edu.sa

In this work, generalized Sobolev spaces are considered. Existence of solutions for strongly nonlinear equation of infinite order of the form $Au + g(x, u) = f$ is established. Here A is an operator from a Sobolev space type to its dual and $g(x, s)$ is a lower order term satisfying a sign condition on s. We consider the case where the data f belongs to L^1.

Keywords: Strongly nonlinear problem; Anisotropic equations; Infinite order; Existence of solutions.

1. Introduction

Let $\Omega \subset I\!R^N$, $N \geq 1$, be a bounded domain, $a_\alpha \geq 0$ and $p_\alpha > 1$ are real numbers for all multi-indices α. The purpose of this paper is to study some anisotropic strongly nonlinear elliptic equations of infinite order, with the data f is in $L^1(\Omega)$. Let A an operator of infinite order defined by

$$Au = \sum_{|\alpha|=0}^{\infty} (-1)^{|\alpha|} D^\alpha \big(a_\alpha |D^\alpha u|^{p_\alpha - 2} D^\alpha u\big).$$

Along this paper we will deal with the following Dirichlet problem of the form

$$Au + g(x,u) = f \quad x \in \Omega. \tag{1.1}$$

Here g is a nonlinear term which has to fulfil a sign condition. If A is a Leray–Lions operator, let us mention that in the isotropic case, several studies have been devoted to the investigation of related problems and a lot of papers have appeared (cf. Benkirane,[2] Benkirane et al.[4] and Brézis et al.[7]). In the anisotropic case, it would be interesting to refer the reader to the works Boccardo et al.,[5] Feng-Quan,[13] also to the recent works Benkirane et al.[3] and Chrif et al.,[9] where the authors proved the existence of solutions of some anisotropic elliptic equations of higher order. For problems of infinite order, let us point out that in this direction Dubinskii[10] proved, under hypothesis $(B_1 - B_4)$ (see Remark 4.1) and certain monotonicity conditions, the existence of solutions for the Dirichlet problem associated with the equation $Au = f$ in some functional Sobolev spaces of infinite order. Our purpose is to prove the same result for strongly nonlinear equations of infinite order of the form (1.1).

2. Preliminaries

Let Ω be a bounded domain in $I\!R^N$. Further $a_\alpha \geq 0, p_\alpha > 1$ are real numbers for all multi-indices α, and $\|.\|_{p_\alpha}$ is the usual norm in the Lebesgue space $L^{p_\alpha}(\Omega)$. For a positive integer m, we define the following vector of real numbers

$$\vec{p} = \{p_\alpha, |\alpha| \leq m\},$$

and denote $\underline{\mathrm{p}} = \min\{p_\alpha, |\alpha| \leq m\}$.
Now, let us consider the generalized functional Sobolev space

$$W^{m,\vec{p}}(\Omega) = \{u \in L^{p_0}(\Omega), D^\alpha u \in L^{p_\alpha}(\Omega), |\alpha| \leq m\}$$

equipped with the norm

$$\|u\| = \sum_{|\alpha|=0}^{m} \|D^\alpha u\|_{p_\alpha}. \tag{2.1}$$

We define the space $W_0^{m,\vec{p}}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ in $W^{m,\vec{p}}(\Omega)$ with respect to the norm (2.1). Note that $C_0^\infty(\Omega)$ is dense in $W_0^{m,\vec{p}}(\Omega)$. Both of $W_0^{m,\vec{p}}(\Omega)$ and $W_0^{m,\vec{p}}(\Omega)$ are reflexive, separable Banach spaces if $p_\alpha > 1$ for all $|\alpha| \leq m$ (the proof of this is an adaptation from R. Adams[1]). $W^{-m,\vec{p}'}(\Omega)$

designs its dual where $\vec{p'}$ is the conjugate of $\vec{p}$ i.e. $p'_\alpha = \frac{p_\alpha}{p_\alpha - 1}$ for all $|\alpha| \le m$.
The Sobolev space of infinite order is the functional space defined by

$$W^\infty(a_\alpha, p_\alpha)(\Omega) = \left\{ u \in C^\infty(\Omega) : \rho(u) = \sum_{|\alpha|=0}^{\infty} a_\alpha \|D^\alpha(u)\|_{p_\alpha}^{p_\alpha} < \infty \right\}.$$

We denote by $C_0^\infty(\Omega)$ the space of all functions with compact support in Ω with continuous derivatives of arbitrary order.
Since we shall deal with the Dirichlet problem, we shall use the functional space $W_0^\infty(a_\alpha, p_\alpha)(\Omega)$ defined by

$$W_0^\infty(a_\alpha, p_\alpha)(\Omega) = \left\{ u \in C_0^\infty(\Omega) : \rho(u) = \sum_{|\alpha|=0}^{\infty} a_\alpha \|D^\alpha u\|_{p_\alpha}^{p_\alpha} < \infty \right\}.$$

In contrast with the finite order Sobolev space, the very first question, which arises in the study of the spaces $W_0^\infty(a_\alpha, p_\alpha)(\Omega)$, is the question of their nontriviality (or nonemptiness), i.e. the question of the existence of a function u such that $\rho(u) < \infty$.

Definition 2.1. (Dubinskii[10]) The space $W_0^\infty(a_\alpha, p_\alpha)(\Omega)$ is called nontrivial space if it contains at least one function which not identically equal to zero, i.e. there is a function $u \in C_0^\infty(\Omega)$ such that $\rho(u) < \infty$.

It turns out that the answer of this question depends not only on the given parameters a_α, p_α of the spaces $W^\infty(a_\alpha, p_\alpha)(\Omega)$, but also on the domain Ω.
The dual space of $W_0^\infty(a_\alpha, p_\alpha)(\Omega)$ is defined as follows

$$W^{-\infty}(a_\alpha, p'_\alpha)(\Omega) = \{ h = \sum_{|\alpha|=0}^{\infty} (-1)^{|\alpha|} D^\alpha h_\alpha,\ \rho'(h) = \sum_{|\alpha|=0}^{\infty} a_\alpha \|h_\alpha\|_{p'_\alpha}^{p'_\alpha} < \infty \},$$

where $h_\alpha \in L^{p'_\alpha}(\Omega)$ and p'_α is the conjugate of p_α, i.e., $p'_\alpha = \frac{p_\alpha}{p_\alpha - 1}$ (for more details about these spaces, see Dubinskii[10] and Dubinskii[11]). By the definition, the duality of the space $W^{-\infty}(a_\alpha, p'_\alpha)(\Omega)$ and $W_0^\infty(a_\alpha, p_\alpha)(\Omega)$ is given by the relation

$$\langle h, v \rangle = \sum_{|\alpha|=0}^{\infty} a_\alpha \int_\Omega h_\alpha(x)\, D^\alpha v(x)\, dx,$$

which, as it is not difficult to verify, is correct.
Let s be a real positive number, we denote by $E(s)$ the integer part of s and denote $\underline{p} = \min\{p_\alpha, |\alpha| \le m\}$. We need the anisotropic Sobolev embedding result.

Lemma 2.1. *Let Ω be a bounded open subset of $I\!R^N$.*
If $m\underline{p} < N$ then $W_0^{m,\vec{p}}(\Omega) \subset L^q(\Omega)$ $\forall q \in [\underline{p}, p^*[$ with $\frac{1}{p^*} = \frac{1}{\underline{p}} - \frac{m}{N}$.
If $m\underline{p} = N$ then $W_0^{m,\vec{p}}(\Omega) \subset L^q(\Omega)$ $\forall q \in [\underline{p}, +\infty[$.
If $m\underline{p} > N$ then $W_0^{m,\vec{p}}(\Omega) \subset L^\infty(\Omega) \cap C^k(\overline{\Omega})$ where $k = E(m - \frac{N}{\underline{p}})$.
Moreover, the embeddings are compacts.

The proof follows immediately from the corresponding embedding theorems in the isotropic case by using the fact that $W^{m,\vec{p}}(\Omega) \subset W^{m,\underline{p}}(\Omega)$.

3. Main result

Let $\Omega \in I\!R^N$ be a bounded domain. We consider the following strongly nonlinear elliptic equation of infinite order of Dirichlet type

$$\sum_{|\alpha|=0}^{\infty} (-1)^{|\alpha|} D^\alpha \big(a_\alpha |D^\alpha u|^{p_\alpha - 2} D^\alpha u\big) + g(x,u) = f \quad \text{in} \quad \Omega, \tag{3.1}$$

where $a_\alpha \geq 0$ and $p_\alpha > 1$ are any sequence of real numbers. Let us define the corresponding function Sobolev space of infinite order by

$$W_0^\infty(a_\alpha, p_\alpha)(\Omega) = \Big\{ u \in C_0^\infty(\Omega) : \rho(u) = \sum_{|\alpha|=0}^{\infty} a_\alpha \|D^\alpha u\|_{p_\alpha}^{p_\alpha} < \infty \Big\}.$$

We assume the following assumptions

(A_1) For all α, $a_\alpha \geq 0$ and $p_\alpha > 1$, moreover the sequence (p_α) is bounded
(A_2) The space $W_0^\infty(a_\alpha, p_\alpha)(\Omega)$ is nontrivial.
(G_1) The function $g : \Omega \times I\!R \mapsto I\!R$ is of Carathéodory, that is, it is measurable in x for each fixed $u \in R$ and continuous in u for almost all $x \in \Omega$, such that, for all $\delta > 0$,

$$\sup_{|u|<\delta} |g(x,u)| \leq h_\delta(x) \in L^1(\Omega).$$

(G_2) The function g satisfies the "sign condition", i.e. $g(x,u)u \geq 0$, for a.e. $x \in \Omega$ and all $u \in I\!R$.

Remark 3.1. In the setting of anisotropic space of finite order and when the operator A is defined by

$$A(u) = \sum_{|\alpha|=0}^{m} (-1)^{|\alpha|} D^\alpha A_\alpha(x, D^\gamma u) \quad |\gamma| \leq |\alpha|,$$

the authors have shown the existence of the solution of the problem (1.1), under the following conditions :

(A$_0$) $A : W_0^{m,\vec{p}}(\Omega) \mapsto W^{-m,\vec{p'}}(\Omega)$ is a bounded operator, pseudo-monotone and coercive, i.e.,

$$\lim_{\|u\|\to+\infty} \frac{\langle Au, u\rangle}{\|u\|} = +\infty,$$

$p_\alpha > 1$, for all $|\alpha| \le m$.

(G$_0$) $g : \Omega \times I\!R \mapsto I\!R$ satisfies the Carathéodory conditions, that is, it is measurable in x for each fixed $u \in R$ and continuous in u for almost all $x \in \Omega$ such that

$$\sup_{|u|<s} |g(x,u)| \le h_s(x),$$

for a.e. $x \in \Omega$, all $s > 0$ and some function $h_s \in L^1(\Omega)$. We assume also the "sign condition" $g(x,u)u \ge 0$, for a.e. $x \in \Omega$ and for all $u \in I\!R$.

We present the following result established in Benkirane,[3] which will be mainly used in the proof of our result.

Theorem 3.1. *(Benkirane et al.[3]) Let $m \in I\!N^*$ such that $m\underline{p} > N$. Suppose (A$_0$) and (G$_0$) are satisfied. Then for all $f \in W^{-m,\vec{p'}}(\Omega)$, there exists $u \in W_0^{m,\vec{p}}(\Omega)$ such that*

$$\begin{cases} g(x,u) \in L^1(\Omega), \ g(x,u)u \in L^1(\Omega) \\ \langle Au, v\rangle + \int_\Omega g(x,u)v\,dx = \langle f, v\rangle, \quad \forall v \in W_0^{m,\vec{p}}(\Omega). \end{cases}$$

Now we shall prove our main result.

Theorem 3.2. *Let us assume the conditions* (A$_1$), (A$_2$), (G$_1$) *and* (G$_2$). *Then for all $f \in L^1(\Omega)$, there exists $u \in W_0^\infty(a_\alpha, p_\alpha)(\Omega)$ such that*

$$\begin{cases} g(x,u) \in L^1(\Omega), g(x,u)u \in L^1(\Omega) \\ \langle Au, v\rangle + \int_\Omega g(x,u)v dx = \langle f, v\rangle \quad for\ all\ v \in W_0^\infty(a_\alpha, p_\alpha)(\Omega). \end{cases}$$

Proof.

In order to get our result, we will deal with the following steps

1. We prove the existence of approximate solutions u_n.
2. We establish the a priori estimates.
3. We prove that u_n converges to an element $u \in W_0^\infty(a_\alpha, p_\alpha)(\Omega)$ and we finally show that u is the solution of our problem.

`Step (1):` The approximate problem.
Let $\varphi \in C_0^\infty(I\!R^N)$, such that, $0 \leq \varphi(x) \leq 1$ and $\varphi(x) = 1$ for x close to 0. Set

$$f_n(x) = \varphi(\frac{x}{n})T_n f(x),$$

with the usual truncation T_n given by

$$T_n\xi = \begin{cases} \xi & \text{if } |\xi| < n \\ \frac{n\xi}{|\xi|} & \text{if } |\xi| \geq n. \end{cases}$$

It is clear that $|f_n| \leq n$ for a.e $x \in \Omega$. Thus, it follows that $f_n \in L^\infty(\Omega)$. Using Lebesgue's dominated convergence theorem, since $f_n \to f$ a.e. $x \in \Omega$ and $|f_n| \leq |f| \in L^1(\Omega)$; we conclude that $f_n \to f$ strongly in $L^1(\Omega)$.
Define the operator of order $2n$ by

$$A_{2n}(u) = \sum_{|\alpha|=0}^{n} (-1)^{|\alpha|} D^\alpha \big(a_\alpha |D^\alpha u|^{p_\alpha - 2} D^\alpha u\big),$$

the operator A_{2n} is clearly monotone and satisfies the growth and the coerciveness conditions. Thanks to Theorem 3.1 (Benkirane et al.[3]), there exists at least one solution $u_n \in W_0^{n,\vec{p}}(\Omega)$ of the following problem

$$(P_n) \begin{cases} g(x,u_n) \in L^1(\Omega) \; and \; g(x,u_n)u_n \in L^1(\Omega) \\ \langle A_{2n}(u_n), v\rangle + \int_\Omega g(x,u_n)v\,dx = \langle f_n, v\rangle \quad \forall v \in W_0^{n,\vec{p}}(\Omega). \end{cases}$$

`Step (2):` A priori estimates.
Set $v = u_n$ and using (A_3), (G_2) and the Hölder inequality, we deduce the estimates

$$\sum_{|\alpha|=0}^{n} a_\alpha \|D^\alpha u_n\|_{p_\alpha}^{p_\alpha} \leq K \tag{3.2}$$

and

$$\int_\Omega g(x,u_n)u_n\,dx \leq K \tag{3.3}$$

for some constant $K = K(f) > 0$. Consequently, we have

$$\|u_n\|_{W_0^{n,\vec{p}}} \leq K. \tag{3.4}$$

Then via a diagonalization process, there exists a subsequence still, denoted by u_n, which converges uniformly to an element $u \in C_0^\infty(\Omega)$, also for all derivatives there holds $D^\alpha u_n \to D^\alpha u$ uniformly in Ω (for more details we refer to Dubinskii[10]).
`Step (3):` Convergence of problem (P_n).

There exists a solution u_n of problem (P_n), $n = 1, 2, \ldots$. Then by passing to the limit, we have

$$\lim_{n\to+\infty} \langle A_{2n}(u_n), v\rangle + \lim_{n\to+\infty} \int_\Omega g(x, u_n)v\, dx = \lim_{n\to+\infty} \langle f_n, v\rangle,$$

for all $v \in W_0^\infty(a_\alpha, p_\alpha)(\Omega)$. Since $f_n \to f$ strongly in $L^1(\Omega)$, it is clear that

$$\lim_{n\to+\infty} \langle f_n, v\rangle = \langle f, v\rangle \quad \text{for all} \quad v \in W_0^\infty(a_\alpha, p_\alpha)(\Omega).$$

Now, we shall prove that

$$\lim_{n\to+\infty} \langle A_{2n}(u_n), v\rangle = \langle Au, v\rangle, \quad \text{for all } v \in W_0^\infty(a_\alpha, p_\alpha)(\Omega).$$

Indeed, let n_0 be a fix number sufficiently large $(n > n_0)$ and let $v \in W_0^\infty(a_\alpha, p_\alpha)(\Omega)$. Set

$$\langle A(u) - A_{2n}(u_n), v\rangle = I_1 + I_2 + I_3,$$

where

$$I_1 = \sum_{|\alpha|=0}^{n_0} \langle A_\alpha(x, D^\gamma u) - A_\alpha(x, D^\gamma u_n), D^\alpha v\rangle$$

$$I_2 = \sum_{|\alpha|=n_0+1}^{\infty} \langle A_\alpha(x, D^\gamma u), D^\alpha v\rangle$$

$$I_3 = -\sum_{|\alpha|=n_0+1}^{n} \langle A_\alpha(x, D^\gamma u_n), D^\alpha v\rangle,$$

where $A_\alpha(x, \xi_\gamma) = a_\alpha|\xi_\alpha|^{p_\alpha-2}\xi_\alpha$, $|\gamma| \le |\alpha|$.
The aim is to prove that I_1, I_2 and I_3 tend to 0. On the one hand, since $A_\alpha(x, \xi_\gamma)$ is of Carathéodory type, $I_1 \to 0$, and the term I_2 is the remainder of a convergent series, hence $I_2 \to 0$. On the other hand, for all $\varepsilon > 0$, there holds $k(\varepsilon) > 0$ (see Brézis[6] p. 56) such that

$$\Big|\sum_{|\alpha|=n_0+1}^{n}\langle A_\alpha(x,D^\gamma u_n),D^\alpha v\rangle\Big|\le\sum_{|\alpha|=n_0+1}^{n}|\langle A_\alpha(x,D^\gamma u_n),D^\alpha v\rangle|$$
$$\le c_0\sum_{|\alpha|=n_0+1}^{n}a_\alpha\int_\Omega|D^\alpha u_n|^{p_\alpha-1}|D^\alpha v|\,dx$$
$$\le c_0\sum_{|\alpha|=n_0+1}^{n}a_\alpha\|D^\alpha u_n\|_{p_\alpha}^{p_\alpha-1}\|D^\alpha v\|_{p_\alpha}$$
$$\le\varepsilon c_0\sum_{|\alpha|=n_0+1}^{n}a_\alpha\|D^\alpha u_n\|_{p_\alpha}^{p_\alpha}$$
$$+c_0k(\varepsilon)\sum_{|\alpha|=n_0+1}^{n}a_\alpha\|D^\alpha v\|_{p_\alpha}^{p_\alpha}$$
$$\le\varepsilon c_0K+c_0k(\varepsilon)\sum_{|\alpha|=n_0+1}^{\infty}a_\alpha\|D^\alpha v\|_{p_\alpha}^{p_\alpha},$$

where K is the constant given in the estimate (3.2). Since the sequence (p_α) is bounded, this implies that $\sum_{|\alpha|=n_0+1}^{\infty}a_\alpha\|D^\alpha v\|_{p_\alpha}^{p_\alpha}$ is the remainder of a convergent series, therefore $I_3\to0$ holds. Hence

$$\langle A_{2n}(u_n),v\rangle\to\langle A(u),v\rangle\ \text{ as }\ n\to+\infty\qquad\text{for all}\qquad v\in W_0^\infty(a_\alpha,p_\alpha)(\Omega).$$

Now we prove that

$$\lim_{n\to+\infty}\int_\Omega g(x,u_n)v\,dx=\int_\Omega g(x,u)v\,dx.$$

Indeed, we have $u_n\to u$ uniformly in Ω, hence $g(x,u_n)\to g(x,u)$ for a.e. $x\in\Omega$. In view of the Fatou lemma and (3.3), we obtain

$$\int_\Omega g(x,u)u\,dx\le\lim_{n\to+\infty}\int_\Omega g(x,u_n)u_n\,dx\le K,$$

this implies $g(x,u)u\in L^1(\Omega)$. On the other hand, for all $\delta>0$ we have

$$|g(x,u_n)|\le\sup_{|t|<\delta}|g(x,t)|+\delta^{-1}|g(x,u_n)u_n|\le h_\delta(x)+\delta^{-1}|g(x,u_n)u_n|.$$

If E is a measurable subset of Ω and $\varepsilon>0$, we have

$$\int_E|g(x,u_n)|\,dx\le\int_E h_\delta(x)\,dx+\delta^{-1}K,$$

where K is the constant of (3.3) which is independent of n. For $|E|$ sufficiently small and $\delta = \frac{2K}{\varepsilon}$, we obtain

$$\int_E |g(x, u_n)|\, dx \le \varepsilon.$$

Using Vitali's theorem we get

$$g(x, u_n) \to g(x, u) \ \text{ in } \ L^1(\Omega).$$

Hence it follows that $g(x, u) \in L^1(\Omega)$. By passing to the limit, we obtain

$$\langle Au, v\rangle + \int_\Omega g(x, u)v\, dx = \langle f, v\rangle, \quad \text{for all } v \in W_0^\infty(a_\alpha, p_\alpha)(\Omega).$$

Finally, we conclude that

$$\begin{cases} g(x, u) \in L^1(\Omega), \ g(x, u)u \in L^1(\Omega) \\ \langle Au, v\rangle + \int_\Omega g(x, u)v\, dx = \langle f, v\rangle, \ \text{ for all } \quad v \in W_0^\infty(a_\alpha, p_\alpha)(\Omega). \end{cases}$$

This completes the proof. □

Remark 3.2. Let us point out here that the result in Theorem 3.2, is established with out assuming the regularity of domain, also without any growth restrictions on p_α for all multi-indice α.

Example 3.1. A prototype example of our problem is defined by

$$\sum_{|\alpha|=0}^{\infty} (-1)^{|\alpha|} D^\alpha \big(a_\alpha |D^\alpha u|^{p_\alpha - 2} D^\alpha u\big) + u|u|^r h(x) = f$$

with $r > 0$, $h \in L^1(\Omega), h(x) \ge 0$ a.e. $x \in \Omega$ and $a_\alpha \ge 0$, $p_\alpha > 1$ are real numbers such that the space $W_0^\infty(a_\alpha, p_\alpha)(\Omega)$ is nontrivial.
Since $h \in L^1(\Omega)$ and $h(x) \ge 0$ a.e. $x \in \Omega$, the function

$$g(x, u) = u|u|^r h(x)$$

satisfies the assumptions (G_1) and (G_2).

Example 3.2. The following example of an operator of infinite order is closely inspired from the one used in.[10] Let us consider the operator

$$Au = [cosD]u(x) \qquad x \in I\!R^N, \quad N \ge 2.$$

Formally we have

$$Au(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n!} D^{2n} u(x).$$

The Dirichlet type problem given by

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2n!} D^{2n} u + g(x,u) = f \quad x \in I\!R^N,$$

have a solution in the nontrivial space of infinite order $W_0^{\infty}(\frac{1}{2n!}, 2)(\Omega)$, when f is an element in $L^1(\Omega)$ and g is a function satisfying the assumptions (G_1) and (G_2).

4. Concluding remarks

Remark 4.1. Let us consider a more general problem, when A is an operator of infinite order defined by

$$A(u) = \sum_{|\alpha|=0}^{\infty} (-1)^{|\alpha|} D^{\alpha} A_{\alpha}(x, D^{\gamma} u) \quad |\gamma| \leq |\alpha|,$$

where $A_\alpha : \Omega \times I\!R^{\lambda_\alpha} \mapsto I\!R$ is a real function, with λ_α denotes the number of multi-indices γ such that $|\gamma| \leq |\alpha|$, satisfying the following assumptions:

(B_1) $A_\alpha(x, \xi_\alpha)$ is a Carathéodory function for all $\alpha, |\gamma| \leq |\alpha|$.

(B_2) For a.e. $x \in \Omega$, all $m \in I\!N^*$, all $\xi_\gamma, \eta_\alpha, |\gamma| \leq |\alpha|$ and some constant $c_0 > 0$, we assume that

$$\left| \sum_{|\alpha|=0}^{m} A_\alpha(x, \xi_\gamma)\eta_\alpha \right| \leq c_0 \sum_{|\alpha|=0}^{m} a_\alpha |\xi_\alpha|^{p_\alpha - 1} |\eta_\alpha|,$$

where $a_\alpha \geq 0, p_\alpha > 1$ are reals numbers for all multi-indices α, and for all bounded sequence $(p_\alpha)_\alpha$.

(B_3) There exist constants $c_1 > 0, c_2 \geq 0$ such that for all $m \in I\!N^*$, for all $\xi_\gamma, \xi_\alpha; |\gamma| \leq |\alpha|$, we have

$$\sum_{|\alpha|=0}^{m} A_\alpha(x, \xi_\gamma)\xi_\alpha \geq c_1 \sum_{|\alpha|=0}^{m} a_\alpha |\xi_\alpha|^{p_\alpha} - c_2.$$

(B_4) The space $W_0^{\infty}(a_\alpha, p_\alpha)(\Omega)$ is nontrivial.

As an open problem, we propose to investigate, under assumptions above (B_1)-(B_4), the existence result of problem

$$Au + g(x,u) = f \qquad x \in \Omega,$$

where g is a nonlinear function and f is an element in $L^1(\Omega)$.

Remark 4.2. In the lot of articles, Dubinskii[10–12] considered the Sobolev spaces of infinite order corresponding to boundary value problem for differential equations of infinite order and obtained the solvability of these problems in the case where the coefficients of the equation grow polynomially with respect to the derivatives.

Dyk Van extends the results of Dubinskii to include the case of operators with rapidly or slowly increasing coefficients (see Dyk Van[8]). In their works, Tran Duk Van et al.[16] introduced Sobolev-Orlicz spaces of infinite order and investigated their principal properties. They also established the existence and uniqueness of solutions of some Dirichlet problems for nonlinear differential equations of infinite order. In particular, let Ω a bounded domain in $I\!R^N$, $N \geq 1$, with boundary $\partial\Omega$. Consider the Dirichlet problem defined by

$$(Pb)\begin{cases} Au(x) = \sum_{|\alpha|=0}^{\infty}(-1)^{|\alpha|}D^\alpha A_\alpha(x,...,D^\alpha u) = f(x) \qquad x \in \Omega, \\ D^\omega u(x) = 0,\ x \in \partial\Omega,\ |\omega| = 0, 1, ... \end{cases}$$

Here $A_\alpha : \Omega \times I\!R^{\lambda_\alpha} \mapsto I\!R$ is a real function, λ_α denotes the number of multi-indices γ such that $|\gamma| \leq |\alpha|$, satisfying the following assumptions:

(H_1) There exist an $N-$function ϕ_α, a function $a_\alpha \in L\{\overline{\phi}_\alpha, \Omega\}$, a continuous bounded function c_α^1, $(1 \leq c_\alpha^1(|t|) \leq const)$ and a constant $b > 0$ such that

$$|A_\alpha(x,\xi)| \leq a_\alpha(x) + b\overline{\phi}_\alpha^{-1}\overline{\phi}_\alpha(c_\alpha^1(|\xi_\alpha|)\xi_\alpha)$$

where

$$\sum_{|\alpha|=0}^{\infty} \|a_\alpha\|_{\phi_\alpha} < +\infty.$$

(H_2) There exist functions $b_m \in L_1(\Omega)$, $g_\alpha \in E\{\phi_\alpha, \Omega\}$, a continuous bounded function c_α^2, $(c_\alpha^2(|t|) \geq c_\alpha^1(|t|))$ and a constant $d > 0$ such that

$$\sum_{|\alpha|=m} (A_\alpha(x,\xi) - g_\alpha(x))\xi_\alpha \geq d \sum_{|\alpha|=m} \phi_\alpha(c_\alpha^2(|\xi_\alpha|)\xi_\alpha) - b_m(x),$$

where

$$\sum_{|\alpha|=0}^{\infty} \|g_\alpha\|_{\phi_\alpha} < +\infty \quad \text{and} \quad \sum_{|\alpha|=0}^{\infty} \int_\Omega |b_m(x)|\,dx < +\infty.$$

(H_3) The $N-$functions ϕ_α are such that the Sobolev-Orlicz space $LW_0^\infty(\phi_\alpha, \Omega)$ is nontrivial.

(H$_4$) For all $\xi = (\xi_0, ..., \xi_\alpha)$ and $\xi' = (\xi'_0, ..., \xi'_\alpha)$ such that $\xi \neq \xi'$ we have the inequality

$$\sum_{|\alpha|=0}^{m} (A_\alpha(x,\xi) - A_\alpha(x,\xi'))(\xi_\alpha - \xi'_\alpha) \geq 0.$$

The corresponding functional setting is the Sobolev-Orlicz of infinite order given by

$$LW_0^\infty(\phi_\alpha, \Omega) = \{u \in C_0^\infty(\Omega) \,:\, \|u\|_\infty < +\infty\},$$

where

$$\|u\|_\infty = \inf\{k > 0 \,:\, \sum_{|\alpha|=0}^{\infty} \int_\Omega \phi_\alpha(\frac{D^\alpha u}{k})\, dx \leq 1\}.$$

The dual space of $LW_0^\infty(\phi_\alpha, \Omega)$ is defined by

$$EW^{-\infty}(\phi_\alpha, \Omega) = \{f \,:\, h(x) = \sum_{|\alpha|=0}^{\infty} (-1)^{|\alpha|} D^\alpha f_\alpha(x)\},$$

where

$$f_\alpha \in E_{\phi_\alpha}(\Omega) \quad \text{ for all multi-indice } \alpha,$$

and

$$\rho'(f) = \sum_{|\alpha|=0}^{\infty} \|f_\alpha\|_{\phi_\alpha} < +\infty.$$

The duality of the spaces $LW_0^\infty(\phi_\alpha, \Omega)$ and $EW^{-\infty}(\phi_\alpha, \Omega)$ is determined by the expression

$$\langle f, v\rangle = \sum_{|\alpha|=0}^{\infty} \int_\Omega f_\alpha(x) D^\alpha v(x)\, dx,$$

which is obviously correct. (For more details about the definition of $N-$function and Sobolev-Orlicz spaces of infinite order we refer to Tran Duk Van et al.[16])

When the data f belongs to the dual, under assumptions (H_1)-(H_4) the authors in Tran Duk Van et al.[16] proved the existence and uniqueness of the solution of the nonlinear problem (Pb).

The same approach allows us to deal with boundary value problem (1.1) in the case of Sobolev-Orlicz spaces of infinite order using operators satisfying $(H_1) - (H_4)$. We thus obtain the existence result related to that of Theorem 3.2.

Acknowledgments

The research of S. El. Manouni is supported by Al-Imam University project No. 28/12.

References

1. R. Adams, *Sobolev Spaces, Academic*, New York, 1975.
2. A. Benkirane, Approximations de type Hedberg dans les espaces $W^m L \log L(\Omega)$ et applications, Ann. Fac. Sci. Toulouse Math. (5) 11 (1990)(2), pp. 67-78.
3. A. Benkirane, M. Chrif and S. El Manouni, Existence results for strongly non-linear elliptic equations of infinite order, Z. Anal. Anwend. (J. Anal. Appl.) 26 (2007), pp. 303-312.
4. A. Benkirane and J.P. Gossez, An approximation theorem in higher order Orlicz-Sobolev spaces and applications, Studia Math. 92 (1989), pp. 231-255.
5. L. Boccardo, T. Gallouet, and P. Marcellini, Anisotropic equations in L^1, Diff. Int. Equations 9 (1996), pp. 209-212.
6. H. Brézis, *Analyse Fonctionnelle. Théorie et Applications.* Paris: Masson 1986.
7. H. Brézis and F. E. Browder, Some properties of higher order Sobolev spaces, J. Math. Pures Appl. (9) 61 (1982), pp. 245-259.
8. Chan Dyk Van, Traces of functions from Sobolev-Orlicz classes of infinite order and nonhomogenous value problems for equations with arbitrary non-linearity, Soviet Math. Dokl. 22 (2) (1980), pp. 626-630.
9. M. Chrif and S. El Manouni, On a strongly anisotropic equation with L^1 data, Appl. Anal. 87(7) (2008), pp. 865-871.
10. Ju. A. Dubinskii, *Sobolev Spaces of Infinite Order and Differential Equations*, Teubner-Texte Math. Band 87. Leipzig: Teubner, 1986.
11. Ju. A. Dubinskii, *Sobolev spaces for infinite order and the behavior of solutions of some boundary value problems with unbounded increase of the order of the equation*, Math. USSR-Sb. 27 (1975)(2), pp. 143-162.
12. Ju. A. Dubinskii, On trace of functions in Sobolev spaces of infinite order and non-homogeneous boundary value problems; Math. USSR Sbornik. 34 (51) (1978), pp. 264-644.
13. L. Feng-Quan, Anisotropic elliptic equations in L^m, J. Convex Anal. 8(2)(2001), pp. 417-422.
14. J. P. Gossez, Some approximation properties in Orlicz Sobolev spaces, Studia Math. 74 (1982), pp. 17-24.
15. J. L. Lions, *Quelques Méthodes de Résolution des Problèmes aux Limites Non Linéaires*, Paris: Dunod; Gauthier-Villars, 1969.
16. Tran Duk Van, R. Gorenflo, Le Van Hap; Sobolev-Orlicz spaces of infinite order and nonlinear differential equations. J. Analysis 10 (1990), pp. 231-245.

Some remarks on a sign condition for perturbations of nonlinear problems

A. Benkirane, J. Benouna, and M. Rhoudaf*

Faculté des Sciences Dhar-Mahraz, B.P 1796 Atlas Fès, Morocco
*E-mail: *rhoudafmohamed@gmail.fr*

In this paper, we shall be concerned with the existence result of the quasilinear elliptic equations of the form,

$$Au + g(x, u, \nabla u) = f,$$

where A is a Leray-Lions operator from $W_0^{1,p}(\Omega)$ into its dual. On the nonlinear lower order term $g(x, u, \nabla u)$, we assume that it is a Carathéodory function having natural growth with respect to $|\nabla u|$, but without assuming the sign condition. The main novelty of our work is a new technique based on a Poincaré's inequality. The right hand side f belongs to $W^{-1,p'}(\Omega)$.

Keywords: Quasilinear elliptic equation; Leray-Lions operator; Existence.

1. Introduction

Let Ω be a bounded open subset of $I\!R^N$, $N \geq 2$. Let p be a real number, with $1 < p < +\infty$, and let p' be its conjugate Hölder exponent (i.e. $\frac{1}{p} + \frac{1}{p'} = 1$).

Let us consider the following nonlinear elliptic problem

$$Au + g(x, u, \nabla u) = f. \tag{1}$$

Where A is a Leray-Lions operator from $W_0^{1,p}(\Omega)$ into its dual $W^{-1,p'}(\Omega)$ and $g(x, u, \nabla u)$ is a nonlinearity which satisfies the following growth condition

$$|g(x, s, \xi)| \leq b(|s|)\gamma(x) + h(s)|\xi|^p$$

with $\gamma \in L^{\frac{p}{p-r}}(\Omega), h \in L^1(I\!R), b(|s|) \leq \beta|s|^{r-1}$where $0 \leq r < p$ and $h, b \geq 0$.

More precisely, this paper deals with the existence of solutions to the following problem

$$\begin{cases} u \in W_0^{1,p}(\Omega), \ g(x,u,\nabla u) \in L^1(\Omega), \\ \displaystyle\int_\Omega a(x,u,\nabla u) T_k(u-v)\, dx + \int_\Omega g(x,u,\nabla u) T_k(u-v)\, dx \\ \qquad\qquad \leq \displaystyle\int_\Omega f T_k(u-v)\, dx \\ \forall\ v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega) \end{cases} \tag{2}$$

where $f \in W^{-1,p'}(\Omega)$.

Our principal goal in this paper is to prove the existence result for the problem (2) without assuming any sign condition on g.

Recently Porreta has proved in[5] the existence result for the problem (2) but the result is restricted to $b(.) \equiv 1$.

A different approach (without sign condition) was used in,[2] under the assumption $g(x,s,\xi) = \lambda s - |\xi|^2$ with $\lambda > 0$.

We recall also that the authors used in[2] the methods of lower and upper-solutions. For the case of sign condition, many important works have appeared during these last decades. Namely.[1]

In the literature of the same problems, the sign condition play a crucial role in the proof of main result, to overcame this difficultly we use a new test functions (see lemma below) and a new technique based under inequality of poincaré.

2. Main results

Let A be the nonlinear operator from $W_0^{1,p}(\Omega)$ into its dual $W^{-1,p'}(\Omega)$ defined as

$$Au = -\text{div}(a(x,u,\nabla u))$$

where $a : \Omega \times I\!R \times I\!R^N$ is a Carathéodory function satisfying the following assumptions: (H_1)

$$|a(x,s,\xi)| \leq [k(x) + |s|^{p-1} + |\xi|^{p-1}] \tag{3}$$

$$[a(x,s,\xi) - a(x,s,\eta)](\xi - \eta) > 0. \text{ for all } \ \xi \neq \eta \in I\!R^N, \tag{4}$$

$$a(x,s,\xi)\xi \geq \alpha |\xi|^p \tag{5}$$

where $k(x)$ is a positive function in $L^{p'}(\Omega)$ and α is a positive constant.

Let $g(x,s,\xi)$ be a Carathéodory function satisfying the following assumptions: (H_2)

$$|g(x,s,\xi)| \leq b(|s|)\gamma(x) + h(s)|\xi|^p \tag{6}$$

where

$$b(|s|) \leq \beta|s|^{r-1} \text{ where } 0 \leq r < p, \quad \gamma \in L^{\frac{p}{p-r}}(\Omega) \tag{7}$$

and $h : I\!R \to I\!R^+$ with $h \in L^1(\Omega)$

Lemma 2.1.
Let $\varphi(t) = c' t e^{\int_0^{|t|} \frac{h(|s|)}{\alpha}\, ds}$, *then* $\varphi'(t) - \frac{h(|t|)}{\alpha}|\varphi(t)| \geq c'$.

Proof. We can tucking $c' = 1$ if $\quad t \geq 0$

$$\begin{aligned} \varphi'(t) &= e^{\int_0^{|t|} \frac{h(|s|)}{\alpha}\, ds} + t\frac{h(|t|)}{\alpha} e^{\int_0^{|t|} \frac{h(|s|)}{\alpha}\, ds} \\ &= e^{\int_0^{|t|} \frac{h(|s|)}{\alpha}\, ds} + \frac{h(|t|)}{\alpha}|\varphi(t)| \end{aligned} \tag{8}$$

if $t \leq 0$

$$\begin{aligned} \varphi'(t) &= e^{\int_0^{|t|} \frac{h(|s|)}{\alpha}\, ds} + t\frac{h(|t|)}{\alpha} e^{\int_0^{|t|} \frac{h(|s|)}{\alpha}\, ds} \\ &= e^{\int_0^{|t|} \frac{h(|s|)}{\alpha}\, ds} + \frac{h(|t|)}{\alpha}|\varphi(t)| \end{aligned}$$

which implies that

$$\varphi'(t) - \frac{h(|t|)}{\alpha}|\varphi(t)| = e^{\int_0^{|t|} \frac{h(|s|)}{\alpha}\, ds} \geq 1.$$

Theorem 2.1. *Assume that the assumption* (H_1) *and* (H_2) *hold and let* f *belongs to* $W^{-1,p'}(\Omega)$. *Then, there exists a measurable function* u *solution of the following problem:*

$$(P)\begin{cases} u \in W_0^{1,p}(\Omega), \quad g(x,u,\nabla u) \in L^1(\Omega) \\ \displaystyle\int_\Omega a(x,u,\nabla u)\nabla T_k(u-\varphi)\, dx + \int_\Omega g(x,u,\nabla u) T_k(u-\varphi)\, dx \\ \displaystyle\leq \int_\Omega f T_k(u-\varphi)\, dx, \qquad \forall \varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega) \quad \forall k > 0. \end{cases}$$

Step (2) Approximate problem
Let us consider the sequence of approximate problem

$$(P_n)\begin{cases} u_n \in W_0^{1,p}(\Omega) \\ \displaystyle\int_\Omega a(x,u_n,\nabla u_n)\nabla(u_n - v)\, dx + \int_\Omega g_n(x,u_n,\nabla u_n)(u_n - v)\, dx \\ \displaystyle\qquad\qquad \leq \int_\Omega f_n(u_n - v)\, dx \\ \forall \quad v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega) \end{cases}$$

where

$$g_n(x,s,\xi) = \frac{g(x,s,\xi)}{1 + \frac{1}{n}|g(x,s,\xi)|}. \tag{9}$$

Note that $|g_n(x,s,\xi)| \le |g(x,s,\xi)|$ et $|g_n(x,s,\xi)| \le n$, then for fixed $n \in I\!N$, the approximate problem (P_n) has at least one solution.[4]

Lemma 2.2. *Let u_n be a solution of the problem (P_n), then we have*

$$\left(\int_\Omega |\nabla u_n|^p\right)^{\frac{1}{p}} \le c, \tag{10}$$

where c is a positive constant not depending on n.

Proof. Let $v = u_n - \varphi(u_n)$ where $\varphi(t) = te^{\int_0^{|t|} \frac{h(|s|)}{\alpha}\, ds}$ (the function h appears in (H_2)). Since $v \in W_0^{1,p}(\Omega))$ v is admissible test function in (P_n), then

$$\int_\Omega a(x,u_n,\nabla u_n)\nabla(\varphi(u_n))\, dx + \int_\Omega g_n(x,u_n,\nabla u_n)\varphi(u_n)\, dx \le \int_\Omega f\varphi(u_n)\, dx$$

which implies that,

$$\begin{aligned}\int_\Omega a(x,u_n,\nabla u_n)\nabla u_n\varphi'(u_n)\, dx \le& \int_\Omega b(|u_n|)\gamma(x)|\varphi(u_n)|\, dx \\ &+ \int_\Omega h(|u_n|)|\varphi(u_n)||\nabla u_n|^p\, dx + \int_\Omega |f||\varphi(u_n)|\, dx\end{aligned}$$

since $\varphi' \ge 0$, then

$$\int_\Omega |\nabla u_n|^p(\varphi'(u_n) - \frac{h(|u_n|)}{\alpha}|\varphi(|u_n|)|) \le \beta c_0\int_\Omega |u_n|^r\gamma(x) + c_0\int_\Omega |f||u_n|$$

with $c_0 = e^{\int_0^{+\infty} \frac{h(|s|)}{\alpha}\, ds}$ and by using poincaré inequality and Lemma 2.1, we deduce that

$$\int_\Omega |\nabla u_n|^p \le c\left(\int_\Omega |\nabla u_n|^p\right)^{\frac{r}{p}} + c\left(\int_\Omega |\nabla u_n|^p\right)^{\frac{1}{p}}. \tag{11}$$

Consequently, since $r < p$ we have

$$\left(\int_\Omega |\nabla u_n|^p\right)^{\frac{1}{p}} \le c \tag{12}$$

then, we can extracts a subsequence still denote by u_n such that,

$$u_n \rightharpoonup u \quad \text{weakly in} \quad W_0^{1,p}(\Omega) \tag{13}$$

$$u_n \to u \quad \text{strongly in} \quad L^p(\Omega) \tag{14}$$

this yields, by (3), the existence of a function $h \in \prod_{i=1}^{N} L^{p'}(\Omega)$ such that

$$a(x, u_n, \nabla u_n) \rightharpoonup h \quad \text{weakly in} \quad \prod_{i=1}^{N} L^{p'}(\Omega). \tag{15}$$

Lemma 2.3. *Let u_n be a solution of the problem (P_n), then we have the following assertions:*
Assertion (i)

$$\lim_{j\to\infty} \lim_{n\to\infty} \int_{\{j\le|u_n|\le j+1\}} a(x, u_n, \nabla u_n)\nabla u_n \, dx = 0. \tag{16}$$

Assertion (ii)

$$\begin{aligned}\lim_{j\to\infty} \lim_{n\to\infty} \int_{\Omega} (a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u), \nabla T_k(u)))\\ \nabla(T_k(u_n) - T_k(u))h_j(u_n) = 0.\end{aligned} \tag{17}$$

Assertion (iii)

$$T_k(u_n) \to T_k(u) \quad \textit{strongly in} \quad W_0^{1,p}(\Omega) \quad \textit{as} \quad n \to \infty. \tag{18}$$

Proof assertion (i). Consider the following function $v = u_n - e^{\int_0^{|u_n|} \frac{h(|s|)}{\alpha}\, ds} T_1(u_n - T_j(u_n))^+$, for $j > 1$, then we obtain,

$$\begin{aligned}
&\int_{\Omega} a(x, u_n, \nabla u_n)\nabla u_n \frac{h(|u_n|)}{\alpha} e^{\int_0^{|u_n|} \frac{h(s)}{\alpha}\, ds} T_1(u_n - T_j(u_n))^+ \\
&+ \int_{\Omega} a(x, u_n, \nabla u_n)\nabla T_1(u_n - T_j(u_n))^+ e^{\int_0^{|u_n|} \frac{h(s)}{\alpha}\, ds} \\
&+ \int_{\Omega} g_n(x, u_n, \nabla u_n) e^{\int_0^{|u_n|} \frac{h(s)}{\alpha}\, ds} T_1(u_n - T_j(u_n))^+ \\
&\le \int_{\Omega} f e^{\int_0^{|u_n|} \frac{h(s)}{\alpha}\, ds} T_1(u_n - T_j(u_n))^+.
\end{aligned}$$

From the growth condition (H_2), we have

$$\int_\Omega a(x,u_n,\nabla u_n)\nabla u_n \frac{h(|u_n|)}{\alpha} e^{\int_0^{|u_n|}\frac{h(s)}{\alpha}\,ds} T_1(u_n - T_j(u_n))^+ + \int_\Omega a(x,u_n,\nabla u_n)\nabla T_1(u_n - T_j(u_n))^+ e^{\int_0^{|u_n|}\frac{h(s)}{\alpha}\,ds}$$
$$\leq \int_\Omega |\gamma(x)| b(|u_n|) e^{\int_0^{|u_n|}\frac{h(s)}{\alpha}\,ds} T_1(u_n - T_j(u_n))^+ + \int_\Omega h(|u_n|)|\nabla u_n|^p e^{\int_0^{|u_n|}\frac{h(s)}{\alpha}\,ds} T_1(u_n - T_j(u_n))^+ + \int_\Omega |f| e^{\int_0^{|u_n|}\frac{h(s)}{\alpha}\,ds} T_1(u_n - T_j(u_n))^+$$

which thanks to (5) and $1 \leq e^{\int_0^{|u_n|}\frac{h(s)}{\alpha}\,ds} \leq e^{\int_0^{+\infty}\frac{h(s)}{\alpha}\,ds} = c_0$, gives:

$$\int_\Omega a(x,u_n,\nabla u_n)\nabla T_1(u_n - T_j(u_n))^+ e^{\int_0^{|u_n|}\frac{h(s)}{\alpha}\,ds} \leq c_0 \int_\Omega |\gamma(x)| b(|u_n|) T_1(u_n - T_j(u_n))^+ + c_0 \int_\Omega |f| T_1(u_n - T_j(u_n))^+ \tag{19}$$

which implies that,

$$\int_{\{j \leq u_n \leq j+1\}} a(x,u_n,\nabla u_n)\nabla u_n \leq c_0\beta \int_{\{|u_n|>j\}} |u_n|^{r-1}|\gamma(x)| T_1(u_n - T_j(u_n))^+ + c_0 \int_\Omega |f| T_1(u_n - T_j(u_n))^+ \tag{20}$$

we deduce that

$$\int_{\{j \leq u_n \leq j+1\}} a(x,u_n,\nabla u_n)\nabla u_n \leq c_0\beta \int_\Omega |u_n|^{r-1}|\gamma(x)| T_1(u_n - T_j(u_n))^+ + \int_\Omega |f| T_1(u_n - T_j(u_n))^+$$
$$\leq c_0\beta \left(\int_\Omega |u_n|^p\right)^{\frac{r}{p}} \left(\int_\Omega (\gamma(x)|T_1(u_n - T_j(u_n))^+|)^{\frac{p}{p-r}}\right)^{\frac{p-r}{p}} + \int_\Omega c_0 |f| T_1(u_n - T_j(u_n))^+ \tag{21}$$

by using the inequality poincaré and (10) we get,

$$\int_{\{j \leq u_n \leq j+1\}} a(x,u_n,\nabla u_n)\nabla u_n \leq c \left(\int_\Omega (\gamma(x)|T_1(u_n - T_j(u_n))^+|)^{\frac{p}{p-r}}\right)^{\frac{p-r}{p}} + \int_\Omega c_0 |f| T_1(u_n - T_j(u_n))^+ \tag{22}$$

then, by Lebesgue's theorem the right hand side goes to zero as n and j tends to infinity.

Therefore, passing to the limit first in n, then in j, we obtain from (22)

$$\lim_{j\to\infty}\lim_{n\to\infty}\int_{\{j\le u_n\le j+1\}} a(x,u_n,\nabla u_n)\nabla u_n\,dx = 0 \tag{23}$$

on the other hand, consider the test function $v = u_n + e^{-\int_0^{|u_n|}\frac{h(s)}{\alpha}\,ds}T_1(u_n - T_j(u_n))^-$ in (P_n), then

$$\begin{aligned}
&\int_\Omega a(x,u_n,\nabla u_n)\nabla\left(-e^{-\int_0^{|u_n|}\frac{h(s)}{\alpha}\,ds}T_1(u_n-T_j(u_n))^-\right)\,dx\\
&+\int_\Omega g_n(x,u_n,\nabla u_n)\left(-e^{-\int_0^{|u_n|}\frac{h(s)}{\alpha}\,ds}T_1(u_n-T_j(u_n))^-\right)\,dx\\
&\le\int_\Omega f\left(-e^{-\int_0^{|u_n|}\frac{h(s)}{\alpha}\,ds}T_1(u_n-T_j(u_n))^-\right)\,dx
\end{aligned}$$

which implies according to (H_2) that,

$$\begin{aligned}
&\int_\Omega a(x,u_n,\nabla u_n)\nabla u_n\frac{h(|u_n|)}{\alpha}e^{-\int_0^{|u_n|}\frac{h(s)}{\alpha}\,ds}T_1(u_n-T_j(u_n))^-\,dx\\
&-\int_\Omega a(x,u_n,\nabla u_n)e^{-\int_0^{|u_n|}\frac{h(s)}{\alpha}\,ds}\nabla T_1(u_n-T_j(u_n))^-\,dx\\
&\le\int_\Omega b(|u_n|)e^{-\int_0^{|u_n|}\frac{h(s)}{\alpha}\,ds}T_1(u_n-T_j(u_n))^-\gamma(x)\,dx\\
&+\int_\Omega h(|u_n|)e^{-\int_0^{|u_n|}\frac{h(s)}{\alpha}\,ds}T_1(u_n-T_j(u_n))^-|\nabla u_n|^p\,dx\\
&+\int_\Omega |f|e^{-\int_0^{|u_n|}\frac{h(s)}{\alpha}\,ds}T_1(u_n-T_j(u_n))^-\,dx.
\end{aligned}$$

From (5) and since $0\le e^{-\int_0^{|u_n|}\frac{h(s)}{\alpha}\,ds}\le 1$ it is possible to conclude that,

$$\begin{aligned}
&-\int_\Omega a(x,u_n,\nabla u_n)e^{-\int_0^{|u_n|}\frac{h(s)}{\alpha}\,ds}\nabla T_1(u_n-T_j(u_n))^-\,dx\\
&\quad\le\int_\Omega b(|u_n|)T_1(u_n-T_j(u_n))^-\gamma(x)\,dx+\int_\Omega |f|T_1(u_n-T_j(u_n))^-\,dx
\end{aligned}$$

then,

$$\begin{aligned}
&-\int_{\{-j-1\le u_n\le -j\}} a(x,u_n,\nabla u_n)\nabla u_n^- e^{-\int_0^{|u_n|}\frac{h(s)}{\alpha}\,ds}\,dx\\
&\qquad\le\int_{\{u_n\le -j\}}\beta|u_n|^{p-1}T_1(u_n-T_j(u_n))^-\gamma(x)\,dx\\
&\qquad\quad+\int_\Omega |f|T_1(u_n-T_j(u_n))^-\,dx
\end{aligned}$$

Consequently, we have

$$\begin{aligned}\int_{\{-j-1\le u_n\le -j\}} & a(x,u_n,\nabla u_n)\nabla u_n \\ & \le \beta\left(\int_{\{u_n\le -j\}}|u_n|^p\right)^{\frac{r}{p}}\left(\int_{\{u_n\le -j\}}(\gamma(x)T_1(u_n-T_j(u_n))^-)^{\frac{p}{p-r}}\right)^{\frac{p-r}{p}} \\ & \qquad + \int_\Omega |f|T_1(u_n-T_j(u_n))^-\,dx\end{aligned} \tag{24}$$

since $(u_n)_{n\in I\!N}$ is bounded in $L^p(\Omega)$, we deduce by Lebesgue's theorem that the terms of the right-hand side of the last inequality goes to zero as n and j tends to infinity. Then, (24) becomes

$$\lim_{j\to\infty}\lim_{n\to\infty}\int_{\{-j-1\le u_n\le -j\}} a(x,u_n,\nabla u_n)\nabla u_n\,dx = 0. \tag{25}$$

Finally, combining (23) and (25), we have

$$\lim_{j\to\infty}\lim_{n\to\infty}\int_{\{j\le |u_n|\le j+1\}} a(x,u_n,\nabla u_n)\nabla u_n\,dx = 0. \tag{26}$$

Proof of assertion (ii). We will use the following function of one real variable, which is defined as follow:

$$h_j(s) = \begin{cases} 1 & \text{if } |s|\le j \\ 0 & \text{if } |s|\ge j+1 \\ j+1-s & \text{if } j\le s\le j+1 \\ s+j+1 & \text{if } -j-1\le s\le -j. \end{cases} \tag{27}$$

Let $v = u_n - e^{\int_0^{|u_n|}\frac{h(s)}{\alpha}\,ds}(T_k(u_n)-T_k(u))^+h_j(u_n)$ as test function in (P_n), we obtain

$$\begin{aligned}&\int_\Omega a(x,u_n,\nabla u_n)\nabla(e^{\int_0^{|u_n|}\frac{h(s)}{\alpha}\,ds}(T_k(u_n)-T_k(u))^+h_j(u_n))\,dx \\ &+\int_\Omega g_n(x,u_n,\nabla u_n)e^{\int_0^{|u_n|}\frac{h(s)}{\alpha}\,ds}(T_k(u_n)-T_k(u))^+h_j(u_n)\,dx \\ &\le \int_\Omega f e^{\int_0^{|u_n|}\frac{h(s)}{\alpha}\,ds}(T_k(u_n)-T_k(u))^+h_j(u_n)\,dx\end{aligned}$$

using (5) and (H_2), we obtain

$$\begin{aligned}&\int_\Omega a(x,u_n,\nabla u_n)\nabla(T_k(u_n)-T_k(u))^+e^{\int_0^{|u_n|}\frac{h(s)}{\alpha}\,ds}h_j(u_n)\,dx\\&\quad\le\int_\Omega b(|u_n|)\gamma(x)e^{\int_0^{|u_n|}\frac{h(s)}{\alpha}\,ds}(T_k(u_n)-T_k(u))^+h_j(u_n)\,dx\\&\quad+\int_{\{j\le|u_n|\le j+1\}}a(x,u_n,\nabla u_n)\nabla u_ne^{\int_0^{|u_n|}\frac{h(s)}{\alpha}\,ds}(T_k(u_n)-T_k(u))^+\,dx\\&\quad+\int_\Omega|f|e^{\int_0^{|u_n|}\frac{h(s)}{\alpha}\,ds}(T_k(u_n)-T_k(u))^+h_j(u_n)\,dx\end{aligned}\tag{28}$$

since $h\in L^1(\Omega)$, we have $e^{\int_0^{|u_n|}\frac{h(s)}{\alpha}\,ds}\le e^{\int_0^{+\infty}\frac{h(s)}{\alpha}\,ds}=c_0<+\infty$, which implies that,

$$\begin{aligned}&\int_\Omega a(x,u_n,\nabla u_n)\nabla(T_k(u_n)-T_k(u))^+e^{\int_0^{|u_n|}\frac{h(s)}{\alpha}\,ds}h_j(u_n)\,dx\\&\quad\le c_0\beta\int_\Omega|u_n|^{r-1}\gamma(x)(T_k(u_n)-T_k(u))^+h_j(u_n)\,dx\\&\quad+c_2\int_{\{j\le|u_n|\le j+1\}}a(x,u_n,\nabla u_n)\nabla u_n\,dx+c_1\int_\Omega|f|(T_k(u_n)\\&\quad-T_k(u))^+h_j(u_n)\,dx\end{aligned}$$

By the Holder's inequality, we have,

$$\begin{aligned}&\int_\Omega a(x,u_n,\nabla u_n)\nabla(T_k(u_n)-T_k(u))^+e^{\int_0^{|u_n|}\frac{h(s)}{\alpha}\,ds}h_j(u_n)\,dx\\&\quad\le c\left(\int_\Omega|u_n|^p\,dx\right)^{\frac{r}{p}}\left(\int_\Omega(\gamma(x)(T_k(u_n)-T_k(u))^+h_j(u_n))^{\frac{p}{p-r}}\,dx\right)^{\frac{p-r}{p}}\\&\quad+c_2\int_{\{j\le|u_n|\le j+1\}}a(x,u_n,\nabla u_n)\nabla u_n\,dx\\&\quad+c_1\int_\Omega|f|(T_k(u_n)-T_k(u))^+h_j(u_n)\,dx\end{aligned}$$

applying again (16) and Lebesgue's theorem the terms of the right hand side of last inequality goes to zero as n and j tend to infinity, then,

$$\lim_{j\to\infty}\lim_{n\to\infty}\int_\Omega a(x,u_n,\nabla u_n)\nabla(T_k(u_n)-T_k(u))^+e^{\int_0^{|u_n|}\frac{h(s)}{\alpha}\,ds}h_j(u_n)\,dx=0.\tag{29}$$

Moreover, (29) becomes,

$$\begin{aligned}&\int_{\{T_k(u_n)-T_k(u)\ge0\}}a(x,T_k(u_n),\nabla T_k(u_n))\nabla(T_k(u_n)-T_k(u))\\&\quad\times e^{\int_0^{|u_n|}\frac{h(s)}{\alpha}\,ds}h_j(u_n)\,dx\\&-\int_{\{T_k(u_n)-T_k(u)\ge0,|u_n|\ge k\}}a(x,u_n,\nabla u_n)\nabla T_k(u)e^{\int_0^{|u_n|}\frac{h(s)}{\alpha}\,ds}h_j(u_n)\,dx\end{aligned}$$

which gives

$$\lim_{j\to\infty}\lim_{n\to\infty}\int_{\{T_k(u_n)-T_k(u)\geq 0\}} a(x,T_k(u_n),\nabla T_k(u_n))(\nabla T_k(u_n)-\nabla T_k(u)) \times e^{\int_0^{|u_n|}\frac{h(s)}{\alpha}\,ds} h_j(u_n)\,dx = 0.$$

Since $e^{\int_0^{|u_n|}\frac{h(s)}{\alpha}\,ds} \geq 1$, the consequently we can write ,

$$\lim_{j\to\infty}\lim_{n\to\infty}\int_{\{T_k(u_n)-T_k(u)\geq 0\}} [a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u))] \times[\nabla T_k(u_n)-\nabla T_k(u)]h_j(u_n)\,dx = 0 \tag{30}$$

on the other hand, taking

$$v = u_n + e^{-\int_0^{|u_n|}\frac{h(s)}{\alpha}\,ds}(T_k(u_n)-T_k(u))^- h_j(u_n)$$

as test function in (P_n) and reasoning as in (30) it is possible to conclude that,

$$\lim_{j\to\infty}\lim_{n\to\infty}\int_{\{T_k(u_n)-T_k(u)\leq 0\}} [a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u))] \times[\nabla T_k(u_n)-\nabla T_k(u)]h_j(u_n)\,dx = 0 \tag{31}$$

Combining (30) and (31), we deduce (17).
Proof of assertion (iii). First we have

$$\begin{aligned}
&\int_\Omega [a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u))][\nabla T_k(u_n)-\nabla T_k(u)]\,dx\\
&= \int_\Omega [a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u))]\\
&\quad\times[\nabla T_k(u_n)-\nabla T_k(u)]h_j(u_n)\,dx\\
&+ \int_\Omega [a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u))]\\
&\quad\times[\nabla T_k(u_n)-\nabla T_k(u)](1-h_j(u_n))\,dx
\end{aligned}$$

thanks to (17) the first integral of the right hand side converges to zero as n and j tend to infinity, for the second term, we have for j large enough $(j > k)$

$$\begin{aligned}
&\int_\Omega [a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u))]\\
&\quad\times[\nabla T_k(u_n)-\nabla T_k(u)](1-h_j(u_n))\,dx\\
&= \int_\Omega a(x,T_k(u_n),\nabla T_k(u))\nabla T_k(u)(1-h_j(u_n))\,dx
\end{aligned}$$

this integral converges to zero since $a(x, T_k(u_n), \nabla T_k(u))$ converges to $a(x, T_k(u), \nabla T_k(u))$ strongly in $(L^{p'}(\Omega))^N$ while $\nabla T_k(u)(1 - h_j(u_n))$ converges to zero strongly in $(L^p(\Omega))^N$.
Finally, we conclude that

$$\lim_{n\to\infty} \int_\Omega [a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u))][\nabla T_k(u_n) - \nabla T_k(u)] = 0$$

then Lemma 5 of,[3] implies that,

$$T_k(u_n) \to T_k(u) \quad \text{strongly in} \quad W_0^{1,p}(\Omega). \tag{32}$$

And

$$\nabla u_n \to \nabla u \qquad \text{a. e. in} \quad \Omega.$$

Lemma 2.4. *Let be u_n a solution of (P_n), then*

$$g_n(x, u_n, \nabla u_n) \to g(x, u, \nabla u) \quad \textit{strongly in} \quad L^1(\Omega) \tag{33}$$

Proof. Let $v = u_n + e^{-\int_0^{|u_n|} \frac{h(|s|)}{\alpha}\, ds} \int_{u_n}^0 h(|s|)\chi_{\{s<-h\}}\, ds$ as test function in (P_n), then

$$\begin{aligned}
&\int_\Omega a(x, u_n, \nabla u_n) \nabla \left(-e^{-\int_0^{|u_n|} \frac{h(|s|)}{\alpha}\, ds} \int_{u_n}^0 h(|s|)\chi_{\{s<-h\}}\, ds \right) \\
&\qquad - \int_\Omega g_n(x, u_n, \nabla u_n) e^{-\int_0^{|u_n|} \frac{h(|s|)}{\alpha}\, ds} \int_{u_n}^0 h(|s|)\chi_{\{s<-h\}}\, ds \\
&\leq \int_\Omega |f| e^{-\int_0^{|u_n|} \frac{h(|s|)}{\alpha}\, ds} \int_{u_n}^0 h(|s|)\chi_{\{s<-h\}}\, ds
\end{aligned}$$

which implies that,

$$\begin{aligned}
&\int_\Omega a(x, u_n, \nabla u_n) \nabla u_n \frac{h(|u_n|)}{\alpha} e^{-\int_0^{|u_n|} \frac{h(|s|)}{\alpha}\, ds} \int_{u_n}^0 h(|s|)\chi_{\{s<-h\}}\, ds \\
&\qquad + \int_\Omega a(x, u_n, \nabla u_n) \nabla u_n e^{-\int_0^{|u_n|} \frac{h(|s|)}{\alpha}\, ds} h(|u_n|)\chi_{\{u_n<-h\}}\, dx \\
&\leq \int_\Omega b(|u_n|)|\gamma(x)| e^{-\int_0^{|u_n|} \frac{h(|s|)}{\alpha}\, ds} \int_{u_n}^0 h(|s|)\chi_{\{s<-h\}}\, ds \\
&\qquad + \int_\Omega h(|u_n|)|\nabla u_n|^p e^{-\int_0^{|u_n|} \frac{h(|s|)}{\alpha}\, ds} \int_{u_n}^0 h(|s|)\chi_{\{s<-h\}}\, ds \\
&\qquad + \int_\Omega |f| e^{-\int_0^{|u_n|} \frac{h(|s|)}{\alpha}\, ds} \int_{u_n}^0 h(|s|)\chi_{\{s<-h\}}\, ds
\end{aligned}$$

using (5) and since $\int_{u_n}^{0} h(|s|)\chi_{\{s<-h\}}\, ds \le \int_{-\infty}^{-h} h(|s|)\, ds$, we get

$$\begin{aligned}\int_\Omega a(x,u_n,\nabla u_n)\nabla u_n e^{-\int_0^{|u_n|}\frac{h(|s|)}{\alpha}\,ds} h(|u_n|)\chi_{\{u_n<-h\}}\, dx \\ \le c\int_\Omega |u_n|^{r-1}\gamma(x)\int_{-\infty}^{-h} h(|s|)\, ds + c\int_\Omega |f| \int_{-\infty}^{-h} h(|s|)\, ds\, dx \\ \le c\int_{-\infty}^{-h} h(|s|)\, ds\Big(\int_{|u_n|\ge 1} |u_n|^{r-1}\gamma(x)\, dx \\ + \int_{|u_n|\le 1} |u_n|^{r-1}\gamma(x)\, dx + c\int_\Omega |f|\, dx\Big) \\ \le c\int_{-\infty}^{-h} h(|s|)\, ds\,(\|\nabla u_n\|_{1,p}{}^r + c' + \|f\|_{1,p'}) \\ \le c\int_{-\infty}^{-h} h(|s|)\, ds\end{aligned}$$

using again (5), we obtain

$$\int_{\{u_n<-h\}} h(|u_n|)|\nabla u_n|^p \le c\int_{-\infty}^{-h} h(|s|)\, ds$$

and since $h \in L^1(I\!R)$, we deduce that,

$$\lim_{h\to+\infty}\sup_{n\in I\!N}\int_{\{u_n<-h\}} h(|u_n|)|\nabla u_n|^p\, dx = 0 \tag{34}$$

consider $v = u_n - e^{\int_0^{|u_n|}\frac{h(|s|)}{\alpha}\,ds}\int_0^{u_n} h(|s|)\chi_{\{s<-h\}}\, ds$ as test function in (P_n), then similarly to (34), we deduce that

$$\lim_{h\to+\infty}\sup_{n\in I\!N}\int_{\{u_n>h\}} h(|u_n|)|\nabla u_n|^p\, dx = 0 \tag{35}$$

assuming (34) and (35) we obtain

$$\lim_{h\to+\infty}\sup_{n\in I\!N}\int_{\{|u_n|>h\}} h(|u_n|)|\nabla u_n|^p\, dx = 0. \tag{36}$$

On the other hand, we have

$$\begin{aligned}\int_{\{u_n>h\}} b(|u_n|)|\gamma(x)|\, dx &\le \int_{\{u_n>h\}} |u_n|^{r-1}|\gamma(x)|\, dx \\ &\le \left(\int_\Omega |u_n|^p\right)^{\frac{r}{p}}\left(\int_{\{u_n>h\}} |\gamma(x)|^{\frac{p}{p-r}}\right)^{\frac{p-r}{p}} \\ &\le c\left(\int_{\{u_n>h\}} |\gamma(x)|^p\right)^{\frac{1}{p}}\end{aligned}$$

then, we conclude that

$$\lim_{h\to+\infty}\sup_{n\in I\!N}\int_\Omega b(|u_n|)|\gamma(x)|\,dx=0 \tag{37}$$

Combining (36), (37) and Vitali's theorem, we conclude (33)

Proof of the theorem. Let $\varphi\in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$ and take $v=u_n+T_k(u_n-\varphi)$ as test function in (P_n), we get

$$\begin{aligned}\int_\Omega a(x,u_n,\nabla u_n)\nabla T_k(u_n-\varphi)\,dx+\int_\Omega g_n(x,u_n,\nabla u_n)T_k(u_n-\varphi)\,dx\\ \le\int_\Omega fT_k(u_n-\varphi)\,dx.\end{aligned} \tag{38}$$

Finally, from (18) and (33), we can pass to the limit in (38), this completes the proof of the theorem.

References

1. A. Bensoussan, L. Boccardo and F. Murat *On a non linear partial differential equation having natural growth terms and unbounded solution,* Ann. Inst. Henri Poincaré 5 N4 (1988) 149-169.
2. L. Boccardo, F. Murat and J.P. Puel, L^∞ *estimate for some nonlinear elliptic partial differential equations and application to an existence result,* SIAM J. Math. Anal. 2 (1992) 326-333.
3. L. Boccardo, F. Murat and J.P. Puel, *Existence of bounded solutions for nonlinear elliptic unilateral problems,* Annali. Mat. Pura Appli. (1988) 183-196.
4. J.L. Lions, *Quelques méthodes de résolution des problèmes aux limites non linéaires,* Dunod, Paris (1969).
5. A. Porretta, *Nonlinear equations with natural growth terms and measure data*, Electronic J. Diff. Equ., Conference 09 (2002) 181-202.

On the principle eigencurve of a coupled system

A. El Khalil

Department of Mathematics, College of Sciences
Al-Imam Muhammad Ibn Saud Islamic
University P.O. Box 90950, Riyadh 11623, Saudi Arabia
E-mail: alakhalil@imamu.edu.sa

We prove that for any $\lambda \in I\!R$, there is a sequence of eigencurves $(\mu_k(\lambda))_k$ for the nonlinear coupled elliptic system

$$\begin{cases} -\Delta_p u = \lambda a(x)|u|^{\alpha-1}|v|^{\beta+1}u + \mu|u|^{\alpha-1}|v|^{\beta+1}u \text{ in } \Omega \\ -\Delta_q v = \lambda a(x)|u|^{\alpha+1}|v|^{\beta-1}v + \mu|u|^{\alpha+1}|v|^{\beta-1}v \text{ in } \Omega \\ (u,v) \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega), \end{cases}$$

by using min-max methods. We prove also via an homogeneity type condition that the eigenvector corresponding to the principal $\mu_1(\lambda)$ is bounded, positive and smooth. We end this work by proving that $\mu_1(\lambda)$ is simple.

Keywords: Quasilinear system; p-Laplacian operator; Principal eigencurve.

1. Introduction

Let Ω be a bounded domain in $I\!R^N$ not necessary regular; $N > 1$, $p > 1$, $q > 1$, and $\alpha\,,\beta > 0$ satisfying the homogeneity assumption:

$$\frac{\alpha+1}{p} + \frac{\beta+1}{q} = 1, \qquad (\mathcal{H})$$

and $a(x) \in L^\infty(\Omega)\backslash\{0\}$ be an indefinite weight function which can change the sign. Here, λ and μ are tow parameter eigenvalues. We consider the following nonlinear elliptic system

$$S_{p,q}(\lambda) \begin{cases} -\Delta_p u = \lambda a(x)|u|^{\alpha-1}|v|^{\beta+1}u + \mu|u|^{\alpha-1}|v|^{\beta+1}u \text{ in } \Omega \\ -\Delta_q v = \lambda a(x)|u|^{\alpha+1}|v|^{\beta-1}v + \mu|u|^{\alpha+1}|v|^{\beta-1}v \text{ in } \Omega \\ (u,v) \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega). \end{cases}$$

Problems where the operator $-\Delta_p$ is present arise both from pure and applied mathematics. Like in the theory of quasiregular and quasiconformal mapping, as well as from a variety of applications, e.g. Non-Newtonian fluids, reaction diffusion problems, flow through porous media. Nonlinear elasticity, glaciology, petroleum extraction, astronomy,... etc.

Many frameworks have been dealing with the corresponding equation

$$-\Delta_p u = \lambda |u|^{\gamma-1} u$$

one can cite for example ANANE,[1] BINDING,[2] THÉLIN et al.,[7] THÉLIN;[6] and in the particular case $p = \gamma$ there are many works, we cite EL KHALIL et al.,[9] FLECKINGER et al.,[10] HESS et al.,[11] KATO,[12] LINDQVIST,[13] RICHARDSON,[14]... etc.

In the system case, we find the work of De Thélin (cf. THÉLIN[6]) where $\lambda = 0$ with Ω is regular bounded domain; also El Khalil, Ouanan and Touzani studied the stability with respect to the rheological exponent p and q of the first eigenvalue see [11], Chabrowski CHABROWSKI[3] considers particular case with $\lambda = 0$ introducing two perturbation functions in the right hand side of the system; and Flekinger and coauthors studied this problem in FLECKINGER et al.[10] with μ is some function on x satisfying some large hypothesis with $\Omega = \mathbb{R}^N$.

In our work, we investigate the situation improving the existence at least a sequence of the eigencurves (eigenpair $(\lambda,\ \mu(\lambda))$ for any bounded domain by using the Ljusternik-Schnirelman theory on C^1-manifolds cf. SZULKIN.[16] So we give a new characterization of the principal eigencurve and also we find some result about the associated eigenvector when λ is in a suitable range of $\mathbb{R}$ depending of $\lambda_1 = \mu_1(0)$ and the weight function $a(x)$.

The rest of this paper is organized as follows. In section 2 we introduce some definitions and prove some technical preliminary, in section 3 we prove that the principal eigencurve is well defined and there exist at least a sequence of the eigencurve, finally we prove some result about the eigenvector associated of $\mu_1(\lambda)$ when λ is in a suitable range.

2. Preliminaries

2.1. *Functional framework*

In the sequel, we shall use the standard notations.

$W_0^{1,p}(\Omega)$ is the completion of $C_0^\infty(\Omega)$ in the space $W^{1,p}(\Omega)$. In the other words,

$$W_0^{1,p}(\Omega) = \left\{ u \in W^{1,p}(\Omega) \mid u_{|\partial\Omega} = 0 \right\}$$

the value of u on Ω being understood in the trace sense.

It is well known that $\left(W_0^{1,p}(\Omega), \|\nabla .\|_p \right)$ is separable, reflexive and uniformly convex, for $0 < p < +\infty$.

The corresponding dual space will be denoted by $W^{-1,p'}(\Omega)$, where $\frac{1}{p}+\frac{1}{p'}=1$.

Remark that for each $u\in W_0^{1,p}(\Omega)$, $\nabla u\in (L^p(\Omega))^N$ and $|\nabla u|^{p-2}\nabla u\in \left(L^{p'}(\Omega)\right)^N$. So, the operator p-Laplacian $-\Delta_p$ may be seen acting from $W_0^{1,p}(\Omega)$ into its dual by

$$\langle -\Delta_p u, w\rangle = \int_\Omega \mid \nabla u\mid^{p-2}\nabla u\nabla w,\ \text{ for } u,w\in W_0^{1,p}(\Omega).$$

It's well know (see e.g. El Khalil et al.[9]) that the inverse of the principal eigenvalue of p-Laplacian is the best constant in the Poincaré's inequality

$$\|u\|_p\le c\|\nabla u\|_p,\ \ \forall u\in W_0^{1,p}(\Omega).$$

we can see also that Δ_p is the unique duality mapping corresponding to the normalization numerical function $f(t)=t^{p-1}$ with respect to the norm $\|\nabla.\|_p$. Consequently the p-Laplacian is an homeomorphism between $W_0^{1,p}(\Omega)$ and $W^{-1,p'}(\Omega)$, with inverse is monotone; bounded and continuous.

2.2. *Definitions*

In this article, all solutions are weak ones, i.e, $(u,v)\in W_0^{1,p}(\Omega)\times W_0^{1,q}(\Omega)$ is a solution of $S_{p,q}(\lambda)$ if for all $(\phi,\psi)\in C_0^\infty(\Omega)\times C_0^\infty(\Omega)$,

$$\begin{cases} \int_\Omega |\nabla u|^{p-2}\nabla u\nabla\phi = \lambda\int_\Omega a(x)|u|^{\alpha-1}|v|^{\beta+1}u\phi + \mu\int_\Omega |u|^{\alpha-1}|v|^{\beta+1}u\phi \\ \\ \int_\Omega |\nabla v|^{q-2}\nabla v\nabla\psi = \lambda\int_\Omega a(x)|u|^{\alpha+1}|v|^{\beta-1}v\psi + \mu\int_\Omega |u|^{\alpha+1}|v|^{\beta-1}v\psi. \end{cases}$$

In order to study the eigenvalue problem $S_{p,q}(\lambda)$, we introduce some functional.

For $(u,v)\in W_0^{1,p}(\Omega)\times W_0^{1,q}(\Omega)$, $\lambda\in I\!R$, we consider

$$A(u,v)=\frac{\alpha+1}{p}\int_\Omega |\nabla u|^p+\frac{\beta+1}{q}\int_\Omega |\nabla v|^q-\lambda\int_\Omega a(x)|u|^{\alpha+1}|v|^{\beta+1};$$

$$B(u,v)=\int_\Omega |u|^{\alpha+1}|v|^{\beta+1}.$$

We set

$$\mathcal{M}=\Big\{(u,v)\in W_0^{1,p}(\Omega)\times W_0^{1,q}(\Omega)/B(u,v)=1\Big\}$$

and

$$\mu_1(\lambda)=inf\left\{A(u,v)/(u,v)\in\mathcal{M}\right\}, \tag{2.1}$$

and for $k \in I\!N^*$

$$\Gamma_k = \{A \subset \mathcal{M} : A \text{ is symmetric, compact, } \gamma(A) = k\},$$

where $\gamma(A)$ is the genus of A, i.e, the smallest integer k such that there exists an odd continuous map from A to $I\!R^k \backslash \{0\}$, see RABINOWITZ[15] and SZULKIN[16] for details of genus.

Definition 2.1. T is said to be in the (S_+) class, if for any sequence u_n weakly convergent in X to u and $\limsup_{n \to +\infty} < Tu_n, u_n - u > \leq 0$, it follows that $u_n \to u$ strongly in$X$.

Definition 2.2. The principal eigencurve of $S_{p,q}(\lambda)$ is the graph of the numerical function $\mu_1 : \lambda \longrightarrow \mu_1(\lambda)$ from $I\!R$ into $I\!R$.

2.3. *Auxiliary lemmas*

Now, we establish the following lemmas that we need in the proof of Theorem 3.1.

Lemma 2.1. *For all $\lambda \in I\!R$, we have*

(i) $\mathcal{M}$ is a closed C^1-manifold.
(ii) A and B are two even functionals and C^1-differentiable in $W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$.
(iii) A is bounded from below in $\mathcal{M}$.

Proof. Recall the following statements:

(i) B is submersion C^1-differentiable and $B'(u,v) \neq 0 \ \forall (u,v) \in \mathcal{M}$ (because $(B'(u,v),(u,v)) = \alpha + \beta + 2 \neq 0$).

(ii) It is obviously.
(iii) By Hölder's inequality we have for all $(u,v) \in \mathcal{M}$

$$\begin{aligned} A(u,v) &= \frac{\alpha+1}{p}\int_\Omega |\nabla u|^p + \frac{\beta+1}{q}\int_\Omega |\nabla v|^q - \lambda \int_\Omega a(x)|u|^{\alpha+1}|v|^{\beta+1} \\ &\geq \frac{\alpha+1}{p}\int_\Omega |\nabla u|^p + \frac{\beta+1}{q}\int_\Omega |\nabla v|^q - |\lambda| \|a\|_\infty B(u,v) \\ &\geq \frac{\alpha+1}{p}\int_\Omega |\nabla u|^p + \frac{\beta+1}{q}\int_\Omega |\nabla v|^q - |\lambda| \|a\|_\infty \\ &\geq \lambda_1 - |\lambda| \|a\|_\infty, \end{aligned} \tag{2.2}$$

where $\lambda_1 = \mu_1(0)$. Thus A is bounded from below. ∎

Lemma 2.2. *For all $\lambda \in I\!R$ we have*

i) *B' is completely continuous in $W_0^{1,p}(\Omega)\times W_0^{1,q}(\Omega)$ and B is continuous for the weakly convergence.*

ii) *A' maps the bounded sets in the bounded sets.*

iii) *If $(u_n,v_n) \rightharpoonup (u,v)$ weakly in $W_0^{1,p}(\Omega)\times W_0^{1,q}(\Omega)$ and $A'(u_n,v_n)$ converges strongly in $(W_0^{1,p}(\Omega)\times W_0^{1,q}(\Omega))'$, then $(u_n,v_n) \longrightarrow (u,v)$ strongly in $W_0^{1,p}(\Omega)\times W_0^{1,q}(\Omega)$.*

iv) *The restriction of A to $\mathcal{M}$ satisfies the Palais-Smale condition i.e, for $(u_n,v_n)\in\mathcal{M}$ if $A(u_n,v_n)$ is bounded and*

$$(A_{|\mathcal{M}})'(u_n,v_n) \longrightarrow 0. \tag{2.3}$$

Then (u_n,v_n) has strong convergent subsequence in $W_0^{1,p}(\Omega)\times W_0^{1,q}(\Omega)$. Here $(W_0^{1,p}(\Omega)\times W_0^{1,q}(\Omega))'$ is the dual space of $W_0^{1,p}(\Omega)\times W_0^{1,q}(\Omega)$ with respect to the standard product norm.

Proof. $i)$ and $ii)$ are obviously.
$iii)$ Let $(u_n,v_n) \rightharpoonup (u,v)$ weakly in $W_0^{1,p}(\Omega)\times W_0^{1,q}(\Omega)$. and $A'(u_n,v_n)$ converges strongly in $(W_0^{1,p}(\Omega)\times W_0^{1,q}(\Omega))'$. Thus $\frac{\partial A}{\partial u}(u_n,v_n)$ $(resp.\frac{\partial A}{\partial v}(u_n,v_n))$ converges strongly in $W^{-1,p'}(\Omega)$ $(resp.W^{-1,q'}(\Omega))$.
On the other hand we have

$$\begin{aligned}< -\Delta_p u_n, u_n - u > &= \tfrac{1}{\alpha+1} < \tfrac{\partial A}{\partial u}(u_n,v_n), u_n - u > \\ &+\lambda < a|u_n|^{\alpha-1}u_n|v_n|^{\beta+1}, u_n - u >\end{aligned}$$

and

$$\begin{aligned}< -\Delta_q v_n, v_n - v > &= \tfrac{1}{\beta+1} < \tfrac{\partial A}{\partial v}(u_n,v_n), v_n - v > \\ &+\lambda < a|u_n|^{\alpha+1}|v_n|^{\beta-1}v_n, v_n - v > .\end{aligned}$$

By passing to the limit we have

$$\limsup_{n\to+\infty} < -\Delta_p u_n, u_n - u > \le 0 \text{ and } \limsup_{n\to+\infty} < -\Delta_q v_n, v_n - v > \le 0.$$

Since p-Laplacian and q-Laplacian satisfy the condition (S^+) we conclude that

$$(u_n,v_n) \longrightarrow (u,v)$$

strongly in $W_0^{1,p}(\Omega)\times W_0^{1,q}(\Omega)$ as $n\to+\infty$.
iv) From (2.3) we have

$$A'(u_n,v_n) - \frac{< A'(u_n,v_n),(u_n,v_n) >}{\alpha+\beta+2} B'(u_n,v_n) \longrightarrow 0 \text{ as } n\to+\infty, \tag{2.4}$$

and thanks to (2.2) we have the boundness of $\{(u_n,v_n)\}_n$ in $\mathcal{M}$.
Therefore, there is a subsequence of (u_n,v_n) still denoted (u_n,v_n), weakly

convergent in $W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$. Moreover, by i) we have $B'(u_n, v_n) \to B'(u,v)$ strongly and $B(u,v) = 1$ (because $B(u_n, v_n) = 1$) , then $(u,v) \in \mathcal{M}$; and by ii) we have $(A'(u_n, v_n))_n$ is bounded.
This and (2.4) implies that $A'(u_n, v_n)$ converges strongly in $(W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega))'$. Finally, by iii), we conclude that $(u_n, v_n) \to (u,v)$ strongly in $W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$. ■

3. Main results

Theorem 3.1. *For any $\lambda \in I\!R$ and any integer $k \in I\!N^*$,*

$$\mu_k(\lambda) = \inf_{A \in \Gamma_k} \max_{(u,v) \in A} A(u,v) \tag{3.1}$$

is critical value of $A(u,v)$ in $\mathcal{M}$. More precisely, there exist $(u_k(\lambda), v_k(\lambda)) \in \mathcal{M}$, $\mu_k(\lambda) \in I\!R$ such that,

$$A(u_k(\lambda), v_k(\lambda)) = \mu_k(\lambda).$$

With $(u_k(\lambda), v_k(\lambda))$ is the eigenfunction of $S_{p,q}(\lambda)$ for the eigenpair eigenvalue $(\lambda, \mu_k(\lambda))$.

Proof. From Szulkin[16] and using Lemma 2.1 and Lemma 2.2 we need only to prove that for all k in $I\!N^*$ $\Gamma_k \neq \emptyset$.
Indeed, $W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$ is separable, thus for all $k \in I\!N^*$ there exist $e_1, e_2, ..., e_k$ k functions of $W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$ linearly dense in $W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$, where $e_i = (e_{i_p}, e_{i_q})$ with $suppe_i \cap suppe_j = \emptyset$ for all $i \neq j$ and $B(e_i) = 1$.
Denote $F_k = span(e_1, e_2, ..., e_k)$ is subspace of $W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$ and $dimF_k = k$; then if $v \in F_k$ there exist $t_1, t_2, ..., t_k$ real numbers such that $v = \Sigma_{i=1}^{i=k} t_i e_i$ thus $B(v) = \Sigma_{i=1}^{i=k} |t_i|^{\alpha+\beta+2}$.
Therefore the map $v \in F_k \longrightarrow B(v)^{\frac{1}{\alpha+\beta+2}}$ is a norm in F_k.
Let $\|(.,.)\|_{p,q}$ the norm in $W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$; hence, there is $c > 0$ such that for all $v \in F_k$, we have

$$c\|v\|_{p,q} \leq B(v)^{\frac{1}{\alpha+\beta+2}} \leq \frac{1}{c}\|v\|_{p,q}.$$

This implies that the set

$$V = F_k \cap \Big\{(u,v) \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega) / B(u,v) \leq 1\Big\} \neq \emptyset$$

is a bounded neighborhood symmetric of $(0,0) \in F_k$. Thus by (f) of Prop. 2.3 of Szulkin,[16] $\gamma(F_k \cap \mathcal{M}) = k$ then $\Gamma_k \neq \emptyset$. ■

Corollary 3.1. *For all* $\lambda \in I\!R$ *we have*

$$\lim_{k\to+\infty} \mu_k(\lambda) = +\infty$$

Proof. $W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$ is separable, hence we can have bi-orthogonal system $(e_k, e_n^*)_n$ such that $e_k \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$, $e_n^* \in (W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega))'$ with $(e_k)_k$ is linearly dense in $W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$; and e_j^* are total in $(W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega))'$, see e.g SZULKIN.[16] Set for $k \in I\!N^*$,

$$F_k = span(e_1, e_2, \ldots, e_k),\ F_k^\perp = \overline{span(e_{k+1}, e_{k+2}, e_{k+3} \ldots)};$$

Therefore, by (g) of proposition 2.3 in SZULKIN,[16] for all $A \in \Gamma_k$ we have $A \cap F_{k-1}^\perp \neq \emptyset\ \forall k \geq n$ hence

$$m_k = \inf_{A\in\Gamma_k} \sup_{A\cap F_{k-1}^\perp} A(u,v) \longrightarrow +\infty \ as\ n \to +\infty \tag{3.2}$$

indeed, if not, for k large enough, there is $(u_k, v_k) \in F_{k-1}^\perp$ such that $B(u_k, v_k) = 1$ and

$$m_k \leq A(u_k, v_k) \leq m,$$

for some constant m independent of k. Thus

$$A(u_k, v_k) = \frac{\alpha+1}{p}\int_\Omega |\nabla u_k|^p + \frac{\beta+1}{q}\int_\Omega |\nabla v_k|^q - \lambda\int_\Omega a(x)|u_k|^{\alpha+1}|v_k|^{\beta+1} \leq m.$$

This implies

$$\frac{\alpha+1}{p}\int_\Omega |\nabla u_k|^p + \frac{\beta+1}{q}\int_\Omega |\nabla v_k|^q \leq m + |\lambda|\|a\|_\infty.$$

Hence (u_k, v_k) is bounded in $W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$. Since $(e_k, e_n^*) = 0\ \forall k \geq n$ and by the Sobolev's imbedding we have $(u_k, v_k) \longrightarrow (0,0)\, as\, n \to +\infty$, strongly in $L^p(\Omega) \times L^q(\Omega)$. This is a contradiction, because $B(u_k, v_k) = 1$. Since $\mu_k(\lambda) \geq m_k$ so we conclude by (3.2) that

$$\lim_{k\to+\infty} \mu_k(\lambda) = +\infty.$$

■

Corollary 3.2. *For all* λ *fixed in* $I\!R$ $\mu_1(\lambda)$ *given by* (2.1) *is the smallest eigenvalue of* $S_{p,q}(\lambda)$ *and there exists* $(u,v) \in \mathcal{M}$ *a solution of* $S_{p,q}(\lambda)$ *associated for* $(\lambda, \mu_1(\lambda))$.

Proof. Let $(u,v) \in M$ and $B = \{(u,v),\ (-u,-v)\}$ therefore $\gamma(B) = 1$ and $B \in \Gamma_1$. Since A is even then $\mu_1(\lambda) \leq \inf_{(u,v)\in M} A(u,v)$, the reverse inequality is obvious. ■

Lemma 3.1.

i) If (u,v) is a solution of $(S_{p,q}(\lambda))$ then $(u,v) \in L^\infty(\Omega) \times L^\infty(\Omega)$.

ii) For all $\lambda \in I\!R$ let (u,v) be an eigenvector associated of $\mu_1(\lambda)$ such that $u \geq 0,\ v \geq 0$ and $B(u,v) = 1$ then: $u > 0\,, v > 0$ and $(u,v) \in C^{1,\eta}_{loc}(\Omega) \times C^{1,\eta}_{loc}(\Omega)$.

Proof. $i)$ deduced from the results of THÉLIN.[6]
$ii)$ Let (u,v) be a solution of $(S_{p,q}(\lambda))$ associated of $(\lambda, \mu(\lambda))$. Thus $(|u|,|v|)$ is also a solution of (2.1). Indeed, $A(|u|,|v|) \leq A(u,v)$ and $B(|u|,|v|) = 1$. Hence, we can suppose that $u \geq 0$ and $v \geq 0$, on the other hand by (i) and Strong Maximum Principle of Vazquez[17] we confirm that $u > 0\,, v > 0$. The results of regularity follow from (i) and local regularity of Dibinedetto.[5]

■

Now, let

$$\Gamma_p(u,\phi) = \int_\Omega |\nabla u|^p + (p-1)\int_\Omega |\nabla \phi|^p \left(\frac{|u|}{\phi}\right)^p - p\int_\Omega |\nabla\phi|^{p-2}\nabla\phi\nabla u\left(\frac{|u|^{p-2}u}{\phi^{p-1}}\right)$$

for all $(u,\phi) \in \left(W^{1,p}_0(\Omega)\right)^2$ with $\phi > 0$ in Ω.

Lemma 3.2. *For all $(u,\phi) \in (W^{1,p}_0(\Omega) \cap C(\Omega))^2$ with $\phi > 0$ in Ω, we have $\Gamma_p(u,\phi) \geq 0$ and if $\Gamma_p(u,\phi) = 0$ there is $c \in I\!R$ such that $u \equiv c\phi$.*

Proof. By Young's inequality, we have for $\epsilon > 0$,

$$\begin{aligned} \nabla u|\nabla\phi|^{p-2}\nabla\phi\tfrac{u|u|^{p-2}}{\phi^{p-1}} &\leq |\nabla u||\nabla\phi|^{p-1}(\tfrac{u}{\phi})^{p-1} \\ &\leq \tfrac{\epsilon^p}{p}|\nabla u|^p + \tfrac{p-1}{p\epsilon^p}|\tfrac{u}{\phi}|^p|\nabla\phi|. \end{aligned} \quad (3.3)$$

For $\epsilon = 1$ we have by integration over Ω,

$$p\int_\Omega |\nabla\phi|^{p-2}\nabla\phi\nabla u(\frac{u}{\phi})^{p-1} \leq \int_\Omega |\nabla u|^p + (p-1)\int_\Omega |\frac{u}{\phi}|^p|\nabla\phi|^p,$$

thus

$$\Gamma_p(u,\phi) \geq 0.$$

On the other hand, if $\Gamma_p(u,\phi) = 0$ by (3.3), we obtain

$$p\int_\Omega |\nabla\phi|^{p-2}\nabla\phi\nabla u(\frac{|u|^{p-2}u}{\phi^{p-1}}) - \int_\Omega |\nabla u|^p - (p-1)\int_\Omega |\frac{u}{\phi}|^p|\nabla\phi|^p = 0 \quad (3.4)$$

and

$$\int_\Omega \left\{\nabla u\nabla\phi|\nabla\phi|^{p-2}\frac{u|u|^{p-2}}{\phi^{p-1}} - |\nabla u||\nabla\phi|^{p-1}(\frac{u}{\phi})^{p-1}\right\}dx = 0. \quad (3.5)$$

By (3.4) we find $|\nabla u| = |\frac{u}{\phi}\nabla\phi|$ thus from (3.5), it follows that $\nabla u = \epsilon\frac{u}{\phi}\nabla\phi$, where $|\epsilon| = 1$. Hence $\Gamma_p(u,\phi) = 0$ implies $\epsilon = 1$ and $\nabla(\frac{u}{\phi}) = 0$. Therefore, there is $c \in I\!R$ such that $u = c\phi$. ■

Theorem 3.2. *For all* $\lambda \in [\frac{-\lambda_1}{2\|a\|_\infty}, \frac{\lambda_1}{2\|a\|_\infty}]$ *, if* $(u(\lambda), v(\lambda))$ *is the eigenvector corresponding to* $\mu_1(\lambda)$*satisfying* $B(u,v) = 1$, $u > 0$, $v > 0$ *then* (u,v) *is unique.*

Proof. Let $(u,v)\,,(\phi,\psi)$ be two positive eigenvectors associated with $\mu_1(\lambda)$ therefore,

$$\begin{cases} -\Delta_p u = \lambda a(x)|u|^{\alpha-1}|v|^{\beta+1}u + \mu|u|^{\alpha-1}|v|^{\beta+1}u & (a)\\ -\Delta_q v = \lambda a(x)|u|^{\alpha+1}|v|^{\beta-1}v + \mu|u|^{\alpha+1}|v|^{\beta-1}v & (b)\end{cases}$$

and

$$\begin{cases} -\Delta_p \phi = \lambda a(x)|\phi|^{\alpha-1}|\psi|^{\beta+1}\phi + \mu|\phi|^{\alpha-1}|\psi|^{\beta+1}\phi & (c)\\ -\Delta_q \psi = \lambda a(x)|\phi|^{\alpha+1}|\psi|^{\beta-1}\psi + \mu|\phi|^{\alpha+1}|\psi|^{\beta-1}\psi. & (d)\end{cases}$$

For any $\epsilon > 0$, let

$$\phi_\epsilon(u,\phi) = \frac{u^p}{(\phi+\epsilon)^{p-1}} \quad \psi_\epsilon(v,\psi) = \frac{v^q}{(\psi+\epsilon)^{q-1}}.$$

Hence, multiplying (a) by u and (c) by $\phi_\epsilon(u,\phi)$, integrating by parts over Ω, and taking the difference, we obtain

$$\Gamma_p(u,\phi) = \int_\Omega (\lambda a(x) + \mu_1)|\phi|^{\alpha+1}|\psi|^{\beta+1}[|\frac{u}{\phi}|^{\alpha+1}|\frac{v}{\psi}|^{\beta+1} - |\frac{u}{\phi}|^p], \qquad (3.6)$$

multiplying (b) by v and (d) by $\psi_\epsilon(v,\psi)$, integrating by parts over Ω and taking the difference, we obtain

$$\Gamma_q(v,\psi) = \int_\Omega (\lambda a(x) + \mu_1)|\phi|^{\alpha+1}|\psi|^{\beta+1}[|\frac{u}{\phi}|^{\alpha+1}|\frac{v}{\psi}|^{\beta+1} - |\frac{v}{\psi}|^p]. \qquad (3.7)$$

So, multiplying (3.6) by $\frac{\alpha+1}{p}$ and (3.7) by $\frac{\beta+1}{q}$, we have

$$\frac{\alpha+1}{p}\Gamma_p(u,\phi) + \frac{\beta+1}{q}\Gamma_q(v,\psi) = \int_\Omega (\lambda a(x)+\mu_1)|\phi|^{\alpha+1}|\psi|^{\beta+1}[|\frac{u}{\phi}|^{\alpha+1}|\frac{v}{\psi}|^{\beta+1}$$

$$-\frac{\alpha+1}{p}|\frac{u}{\phi}|^p - \frac{\beta+1}{q}|\frac{v}{\psi}|^q]. \qquad (3.8)$$

By Young's inequality we have

$$|\frac{u}{\phi}|^{\alpha+1}|\frac{v}{\psi}|^{\beta+1} - \frac{\alpha+1}{p}|\frac{u}{\phi}|^p - \frac{\beta+1}{q}|\frac{v}{\psi}|^q \leq 0.$$

According to Lemma 3.2 and (3.8), we have by using $|\lambda| \leq \frac{\lambda_1}{2\|a\|_\infty}$,

$$0 \leq \frac{\alpha+1}{p}\Gamma_p(u,\phi) + \frac{\beta+1}{q}\Gamma_q(v,\psi) \leq 0.$$

Hence

$$\Gamma_p(u,\phi) = \Gamma_q(v,\psi) = 0.$$

This and Lemma 3.2 imply that $\phi \equiv tu$ and $\psi \equiv t'v$. Then by using the normalization $B(u,v) = 1$ we conclude that $u \equiv \phi$, $v \equiv \psi$. ■

Acknowledgements. The author gratefully acknowledges the financial support provided by Al-Imam Muhammed Ibn Saud Islamic University during this research.

References

1. A. Anane: *Etude des valeurs propres et de la résonance pour l'opérateur p-Laplacien*, thèse de Doctorat, U.L.B (1987-88).
2. P.A. Binding and Y.X. Huang: *The principal eigencurve for p-Laplacian*, Diff. int. Equations, 8, n.2 (1995), 405-415.
3. J. Chabrowski: *On multiple solutions for nonhomogeneous system of elliptic equations*, Revista Matimatica de la Universidad Computense de Madrid volume 9, numero 1, 1996.
4. J.I. Diaz and J.E. Saa: *Existence et unicité de solutions positives pour certaines équations elliptiques quasilinéaires.* C.R. Acad. Sci. Paris, serie I, Math. 305,(1987), 521-524.
5. E. DiBenedetto: *$C^{1+\alpha}$-local regularity of weak solutions of degenerate elliptic equations*, Nonlinear Analysis T.M.A. 7, (1983), 827-859.
6. F. De Thélin: *Première valeur propre d'un système elliptique non linéaire*, C.R. Acad. Sci. Paris,t. 311, Serie I, (1990), 603-606.
7. F. De Thélin and J. Vélin: *Existence et non existence de solutions non triviales pour des systémes elliptiques non-linéaires*, C.R. Acad. Sci. Paris, 313, serie I, (1991), 589-592.
8. A. El Khalil, A. El Manouni and M. Ouanan: *Simplicity and stability of the first eigenvalue of a nonlinear elliptic system*, International Journal of Mathematics and Mathematical Sciences, Vol. (2005)10, (2005), 1555-1563.
9. A. El Khalil and A. Touzani: *On the first eigencurve of the p-Laplacian*, Lecture Note in Pure and Applied Mathematics, Vol. 229 (2002), 195-205.
10. J. Fleckinger, R.F. Manasevich, N.M. Stavrakakis and F. De Thélin: *Principal eigenvalue for some quasilinear elliptic equations on $\mathbb{R}^N$*, Advances in Diffrential Equations, Vol. 2, No.6 (1997), 981-1003.
11. P. Hess and T. Kato: *On some liner and nonliner eigenvalue problems with an indefinite weight function*, Comm.P.D.E. 5 (1980), 999-1030.
12. T. Kato: *Superconvexity of the spectral radius and convexity of the spectral bond and the type*, Math.Z., 180 (1982), 265-273.

13. P. LINDQVIST: *On the equation* $div(|\nabla u|^{p-2}\nabla u)+\lambda|u|^{p-2}u=0$, Proc.Amer. Math. Soc., 109 (1990), 157-164.
14. R.G.D. RICHARDSON: *Theorems of oscillation for tow linear differential equations of the second order with two parameters,* Tran. Amer. Math. SOC., 13(1992), 22-34.
15. P.H. RABINOWITZ: *Minimax methods in critical point theory with applications to differential equations*, AMS Rigional Conference Series in Math. vol. 65 (1986).
16. A. SZULKIN: *Ljusternik-Schnirelmann theory on* C^1*-manifolds*, Ann. Inst. Henri Poincaré, Anal. Non. 5 (1988), 119-139.
17. J.L. VAZQUEZ: *A strong maximum principle for some quasilinear elliptic equations*, Appl. Math. Optim. 12 (1984), 191-202.

Parabolic inclusions with nonlocal conditions

Abdelkader Boucherif

King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics
Box 5046 Dhahran, 31261, Saudi Arabia
E-mail: aboucher@kfupm.edu.sa

In this paper we consider the solvability of nonlinear parabolic differential equations with discontinuous nonlinearities, subjected to nonlocal conditions. We are concerned with the existence of solutions. Our technique is based on the Green's function for linear parabolic partial differential equations, the maximum principle, fixed point theorems for multivalued maps and the method of lower and upper solutions.

Keywords: Parabolic problems; Integral representation of solutions; Maximum principles; Multivalued maps; Nonlocal conditions; Fixed point theorems; Lower and upper solutions.

1. Introduction

Let Ω be an open bounded domain in $\mathbb{R}^N$, $N \geq 2$, with a smooth boundary $\partial\Omega$. Let T be a positive real number, $D = \Omega \times (0, T]$ and $\Gamma = \partial\Omega \times [0, T]$. Given continuous functions $\phi, \beta_i : \Omega \to \mathbb{R}$, $i = 1, 2, ..., m$ and numbers t_i, $i = 1, 2, ..., m$ in $(0, T)$ with $0 < t_1 < ... < t_m < T$, we are concerned with the existence of solutions of the following parabolic problem with a multivalued right-hand side and nonlocal conditions

$$D_t u + Lu \in F(x, t, u) \qquad (x, t) \in D, \tag{1}$$

$$u(x, t) = 0 \qquad (x, t) \in \Gamma, \tag{2}$$

$$u(x, 0) + \sum_{i=1}^{m} \beta_i(x) u(x, t_i) = \phi(x) \quad x \in \Omega, \tag{3}$$

where L is a strongly elliptic operator given by

$$Lu = -\sum_{i,j=1}^{N} a_{ij}(x,t)D_iD_ju + c(x,t)u.$$

Parabolic problems with discontinuous nonlinearities arise in the description of many phenomena in the applied sciences. We can mention, for instance, chemical reactor theory,[15] porous medium combustion,[13],[14] best response dynamics arising in game theory,[10],[18] . Parabolic problems with discontinuous nonlinearities have been also investigated in the papers,[6],[8],[7],[30],[31],[33] . The importance of nonlocal conditions and their applications in different field has been discussed in,[3],[5],[12] . Several papers have dealt with parabolic problems with continuous nonlinearities and nonlocal conditions. See for instance,[20],[21],[25],[26],[34],[35] .

In this paper we consider a nonlocal problem for a class of nonlinear parabolic equations with a multivalued right hand side. We shall convert problem (1), (2), (3), to an integral inclusion using the properties of the Green's function corresponding to the linear problem. We, then, provide sufficient conditions on the data that will enable us to obtain at least one solution. Our approach is based on fixed point theorems for suitable multivalued operators and the method of lower and upper solutions.

The outline of the paper is as follows. In section 2 we introduce notations and function spaces which will be used in the paper. In section 3, we shall study the linear nonhomogeneous problem and the properties of the Green's function. In section 4 we recall the main properties of multivalued maps. We state and prove our main results in section 5.

2. Preliminaries

In this section we introduce some notations and function spaces. Let Ω be an open bounded domain in $\mathbb{R}^N$, $N \geq 2$, with a smooth boundary $\partial\Omega$. Let T be a positive real number, $D = \Omega \times (0,T)$ and $\Gamma = \partial\Omega \times [0,T]$. Then Γ is smooth and any point on Γ satisfies the inside (and outside) strong sphere property, see;[16] i.e. for any $(x_0,t_0) \in \Gamma$ there is a closed ball $B \subset \Omega$ (and a closed ball $\widetilde{B}$ outside Ω) such that $\Gamma \cap (B \times [0,T]) = \{(x_0,t_0)\}$, (and $\Gamma \cap \left(\widetilde{B} \times [0,T]\right) = \{(x_0,t_0)\}$). For $u : D \to \mathbb{R}$ we denote its partial derivatives (when they exists) by $D_tu = \partial u/\partial t$, $D_iu = \partial u/\partial x_i$, $D_iD_ju = \partial^2 u/\partial x_i \partial x_j$, $i,j = 1,...,N$.

$C(\overline{D})$ denotes the Banach space of continuous functions $u : \overline{D} \to \mathbb{R}$, endowed with the norm

$$|u|_0 = \sup\{|u(x,t)|\,;\ (x,t) \in \overline{D}\}.$$

We say that $u \in C^{2,1}(D)$ if u, $D_i u$, $D_i D_j u$ and $D_t u$ exist and are continuous on D. In fact, we can write

$$C^{2,1}(D) = \{u \in C(D);\ u(.,t) \in C^2(\Omega), t \in (0,T)\,,\ u(x,.) \in C^1\,(0,T)\,, x \in \Omega\}.$$

$u \in C(D)$ is called Hölder continuous of order $\alpha \in (0,1]$ if

$$H_\alpha\,(u) = \sup\{\frac{|u(x,t) - u(\xi,\tau)|}{\left(\|x-\xi\|^2 + |t-\tau|^2\right)^{\alpha/2}}; (x,t)\,, (\xi,\tau) \in D\} < +\infty.$$

In this case we write $u \in C^\alpha(D)$ and we define its norm by

$$|u|_\alpha = |u|_0 + H_\alpha\,(u)\,.$$

If $\alpha = 1$, u is called Lipschitz continuous and we write $u \in Lip(D)$. Note that the natural injection $i : C^\alpha(D) \to C(D)$ is continuous. We say that that $C^\alpha(D)$ is continuously embedded in $C(D)$, and we write $C^\alpha(D) \hookrightarrow C(D)$.

Also, $u \in C^{2+\alpha,1+\alpha}(D)$ if $u(.,t) \in C^{2+\alpha}(\Omega)$ for all $t \in (0,T)$ and $u(x,.) \in C^{1+\alpha}(0,T)$ for all $x \in \Omega$. For $u \in C^{2+\alpha,1+\alpha}(D)$ we define its norm by

$$|u|_{2+\alpha,1+\alpha} = |u|_\alpha + \sum_{i=1}^{N} |D_i u|_\alpha + \sum_{i,j=1}^{N} |D_i D_j u|_\alpha + |D_t u|_\alpha\,.$$

We say that $\partial\Omega$ is in the class $C^{\ell+\alpha}, \ell \in \mathbb{N},\ \alpha \in [0,1)$ if in a neighborhood of each point of $\partial\Omega$ there is a local reprentation of $\partial\Omega$ having the form $x_i = \vartheta_i(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_N)$ with $\vartheta_i \in C^{\ell+\alpha}$.

Also, for $1 \le p < +\infty$, we say that $u : D \to \mathbb{R}$ is in $L^p\,(D)$ if u is measurable and $\int_D |u(x,t)|^p\,dxdt < +\infty$, in which case we define its norm by

$$|u|_{L^p} = \left(\int_D |u(x,t)|^p\,dxdt\right)^{1/p}.$$

3. Linear nonhomogeneous problem

In this section we consider the linear nonhomogeneous problem

$$D_t u + Lu = f(x,t), \qquad (x,t) \in D, \tag{4}$$

$$u(x,t) = 0, \qquad (x,t) \in \Gamma, \tag{5}$$

with the following nonlocal initial condition

$$u(x,0)+\sum_{i=1}^{m}\beta_i(x)u(x,t_i)=\phi(x)\,,\quad x\in\Omega. \tag{6}$$

We shall assume throughout this paper that the function $\phi:\Omega\to\mathbb{R}$ is continuous and the functions $a_{ij},c:D\to\mathbb{R}$ are Hölder continuous, $a_{ij}=a_{ji}$ and moreover there exist positive numbers λ_0,λ_1 such that

$$\lambda_0\left\|\xi\right\|^2\le\sum_{i,j=1}^{N}a_{ij}(x,t)\xi_i\xi_j\le\lambda_1\ \left\|\xi\right\|^2,\ \forall\xi\in\mathbb{R}^N \text{ and } \forall\,(x,t)\in D.$$

The classical problem, i.e.when $\beta_i=0$ for all $i=1,2,...,m$, is well known and completely solved (see the books,[16],[22],[24],[28]). Problem (4), (5), (6) has been investigated by several authors (see for instance,[9],[21],[35] and the references therein). The following version of the maximum principle can be found in,[9],[28] .

Lemma 3.1. *Let $u\in C^{2,1}(D)\cap C(\overline{D})$. Assume that $c(x,t)\ge c_0>0$ on D and $\beta_i(x)\le 0$ on Ω with $-1\le\sum_{i=1}^{m}\beta_i(x)\le 0$ on Ω. If $D_tu+Lu\ge 0$ in D, $u(x,t)\ge 0$ on Γ, $u(x,0)+\sum_{i=1}^{m}\beta_i(x)u(x,t_i)\ge 0$ on Ω. Then $u(x,t)\ge 0$ on $\overline{D}$. Moreover, either $u(x,t)=0\ \forall\,(x,t)\in D$, or $u(x,t)>0$ on D.*

Proof. Suppose there exists $(\eta,\tau)\in\overline{D}$ such that $u\,(\eta,\tau)<0$. It follows from the continuity of u that u achieves a negative minimum at some point $(x_0,t_0)\in\overline{D}$. By the strong maximum principle (see[16]) we have either $(x_0,t_0)\in\Gamma$ or $(x_0,t_0)=(x_0,0)$ with $x_0\in\Omega$. From the above assumptions we see that $\min_{\overline{D}}u\,(x,t)=u\,(x_0,0)<0$. Hence, if $1+\sum_{i=1}^{m}\beta_i(x_0)>0$,

$$0\le u(x_0,0)+\sum_{i=1}^{m}\beta_i(x_0)u(x_0,t_i)\le u(x_0,0)\left(1+\sum_{i=1}^{m}\beta_i(x_0)\right)<0,$$

which is a contradiction.

In case $1+\sum_{i=1}^{m}\beta_i(x_0)=0$ we let $u(x_0,t_k)=\min\{u\,(x_0,t_i)\,;\ i=1,2,...,m\}$. Then

$$\begin{aligned}u(x_0,0)-u(x_0,t_k)&=u(x_0,0)+u(x_0,t_k)\sum_{i=1}^{m}\beta_i(x_0)\\&\ge u(x_0,0)+\sum_{i=1}^{m}\beta_i(x_0)u\,(x_0,t_i)\ge 0.\end{aligned}$$

This last inequality shows that u takes on a negative minimum at $(x_0,t_k)\in D$, which is not possible. This completes the proof of the Lemma.

One of the important features of the linear problem is the integral reprentation of solutions.

The homogeneous problem

$$D_t u + Lu = 0, \qquad (x,t) \in D$$

$$u(x,t) = 0, \qquad (x,t) \in \Gamma$$

$$u(x,0) + \sum_{i=1}^{m} \beta_i(x) u(x,t_i) = 0, \quad x \in \Omega$$

has a only the trivial solution, provided that $-1 \leq \sum_{i=1}^{m} \beta_i(x) \leq 0$.

There exists a unique function, $G(x,t;y,s)$, called the Green's function corresponding to the linear homogeneous problem. This function satisfies the following (,[16],[2228]),

(i) $D_t G + LG = \delta(t-s)\,\delta(x-y), \quad s < t,\ x, y \in \Omega$

(ii) $G(x,t;y,s) = 0, \quad s > t, \ x, y \in \Omega$

(iii) $G(x,t;y,s) = G(x,t;y,s) = 0, \quad s < t,\ x, y \in \partial\Omega,$

(iv) $G(x,t;y,s) > 0$ for $(x,t) \in D$

(v) $G,\ D_t G,\ D_i G,\ D_i D_j G$ are continuous functions of $(x,t),\ (y,s) \in D,\ t-s > 0,$

(vi) $|G(x,t;y,s)| \leq C(t-s)^{-N/2} \exp\left(\dfrac{-a\,\|x-y\|^2}{t-s}\right)$, for some positive constants C, a.

Lemma 3.2. *Assume that the function f is Hölder continuous and $\sum_{i=1}^{m} |\beta_i(x)| < 1$. Then, Problems (4), (5), (6) has a unique strong solution, i.e. a solution $u \in C^{2,1}(D) \cap C(\overline{D})$.*

Proof. Consider the following representation (,[9],[1628]),

$$u(x,t) = \int_{\Omega} G(x,t;y,0)\ u(y,0)\,dy + \int_0^t \int_{\Omega} G(x,t;y,s)\ f(y,s)\,dyds \quad (7)$$

for $(x,t) \in D$, where $u(y,0)$ has to be determined. Using condition (6) we see that $u(x,0)$ is a solution of the following Fredholm integral equation of the second kind

$$\begin{cases} u(x,0) + \sum_{i=1}^{m} \beta_i(x) \int_{\Omega} G(x,t_i;y,0)\ u(y,0)\,dy = \\ -\sum_{i=1}^{m} \beta_i(x) \int_0^{t_i} \int_{\Omega} G(x,t_i;y,s)\ f(y,s)\,dyds + \phi(x)\,. \end{cases} \quad (8)$$

The condition $\sum_{i=1}^{m}|\beta_i(x)|<1$ implies that Eq. (8) with $f=0$ and $\phi=0$ has only the trivial solution. Hence there exists a unique solution $u(.,0)$ of (8). Consequently (7) gives the unique solution of (4), (5), (6).
From the above discussion we see that for each $v\in C(\Omega)$ the problem (4), (5) and

$$u(x,0)=v(x) \tag{9}$$

has a unique solution $u\in C^{2,1}(D)\cap C(\overline{D})$, which we denote by $u(x,t;v)$ and has the following representation

$$u(x,t;v)=\int_0^t\int_\Omega G(x,t;y,s)\ f(y,s)\,dyds+\int_\Omega G(x,t;y,0)\ v(y)\,dy$$

for $(x,t)\in D$.

It is clear from the above representation that the functions φ and ψ defined respectively by $\varphi(x,t)=\int_0^t\int_\Omega G(x,t;y,s)\ dyds$ and $\psi(x,t)=\int_\Omega G(x,t;y,0)dy$ are continuous on D. Let $K_0:=\sup_{(x,t)}\int_0^t\int_\Omega G(x,t;y,s)\ dyds$.
Following[26] we define an operator

$$\Upsilon:C(\Omega)\to C(\Omega),$$

by

$$\Upsilon v(x)=-\sum_{i=1}^{m}\beta_i(x)u(x,t_i;v)+\phi(x). \tag{10}$$

Lemma 3.3. *Assume* $max_{x\in\Omega}\{\sum_{i=1}^{m}|\beta_i(x)|\int_\Omega G(x,t_i;y,0)dy\}<1$. *Then* Υ *is a Lipschitz operator.*

Proof. Let $v_1,v_2\in C(\Omega)$ and let $u_1(x,t;v_1)$ and $u_2(x,t;v_2)$ be the corresponding solutions of (4), (5), (9) respectively.

$$(\Upsilon v_1)(x,t)-(\Upsilon v_2)(x,t)=-[\sum_{i=1}^{m}\beta_i(x)u(x,t_i;v_1)-\sum_{i=1}^{m}\beta_i(x)u(x,t_i;v_2)].$$

It follows from (7) that

$$(\Upsilon v_1)(x,t)-(\Upsilon v_2)(x,t)=$$

$$-\sum_{i=1}^{m}\beta_i(x)\int_\Omega G(x,t_i;y,0)\ [v_1(y)-v_2(y)]dy.$$

Thus

$$|(\Upsilon v_1)(x,t) - (\Upsilon v_2)(x,t)|$$
$$\leq \{\sum_{i=1}^{m} |\beta_i(x)| \int_{\Omega} |G(x,t_i;y,0)|\, dy\} \, \|v_1 - v_2\|_0 \, .$$

Therefore

$$\|(\Upsilon v_1) - (\Upsilon v_2)\|_0 \leq max_{x\in\Omega}\{\sum_{i=1}^{m} |\beta_i(x)| \int_{\Omega} |G(x,t_i;y,0)|\, dy\} \, \|v_1 - v_2\|_0 \, .$$

Consequently, Υ has a unique fixed point $v_0 \in C(\Omega)$, and $u = u(x,t;v_0)$ is the unique solution of (4), (5), (6).

4. Multivalued functions

In this section we introduce some useful definitions and properties from set-valued analysis. For complete details on multivalued maps we refer the interested reader to the books,[1],[2],[11] and.[19]

Let $(X, |.|_X)$ and $(Y, |\cdot|_Y)$ be Banach spaces. The domain of a multivalued map $\Re : X \to 2^Y$ is the set $dom\Re = \{z \in X;\ \Re(z) \neq \emptyset\}$. $\Re$ is convex (closed) valued if $\Re(z)$ is convex (closed) for each $z \in X$. $\Re$ is bounded on bounded sets if $\Re(A) = \cup_{z\in A}\Re(z)$ is bounded in Y for all bounded subsets $A \subset X$ (i.e. $\sup_{z\in A}\{\sup\{|y|\, ; y \in \Re(z)\}\} < \infty$). $\Re$ is called upper semicontinuous (usc) on X if for each $z \in X$ the set $\Re(z)$ is nonempty, closed in Y, and for each open subset B of Y containing $\Re(z)$, there exists an open neighborhood A of z in X such that $\Re(A) \subset B$. In terms of sequences, $\Re$ is usc if for each sequence $(z_n) \subset X$, $z_n \to z_0$, and B a closed subset of Y such that $\Re(z_n) \cap B \neq \emptyset$ then $\Re(z_0) \cap B \neq \emptyset$. The set-valued map $\Re$ is called completely continuous if $\Re(A)$ is relatively compact in Y for every bounded subset A of X. If $\Re$ is completely continuous with nonempty compact values, then $\Re$ is usc if and only if $\Re$ has a closed graph (i.e. $z_n \to z$, $w_n \to w$, $w_n \in \Re(z_n) \Rightarrow w \in \Re(z)$).

$\Re$ is called lower semicontinuous (lsc) on X if $\Re^{-1}(B)$ is open in X whenever B is open in Y, or $\{z \in X; \Re(z) \subset B\}$ is closed in X whenever B is closed in Y. It can be shown that $\Re$ is lsc if and only if $d_Y(y, \Re(.))$ is usc for every $y \in Y$.

When $X \subset Y$ then $\Re$ has a fixed point if there exists $z \in X$ such $z \in \Re(z)$. When $D \subset X = \mathbb{R}^{N+1}$ and $Y = \mathbb{R}$ the multivalued map $\Re : \overline{D} \to 2^{\mathbb{R}}$ with closed values is called measurable if for every $\varkappa \in \mathbb{R}$, the function $v \longmapsto dist(\varkappa, \Re(v)) = \inf\{|\varkappa - z|\, ; z \in \Re(v)\}$ is measurable.

Definition 4.1. A multivalued map $F : D \times \mathbb{R} \to 2^{\mathbb{R}} \backslash \emptyset$ is called an L^2- Carathéodory multifunction if

(i) $(x,t) \mapsto F(x,t,u)$ is measurable for each $u \in \mathbb{R}$,

(ii) $u \mapsto F(x,t,u)$ is upper semicontinuous for almost all $(x,t) \in D$,

(iii) for each $r > 0$ there exists $\omega_r \in L^2(D)$ such that $|F(x,t,u)| \leq \omega_r(x,t)$ a.e. on D whenever $|u| \leq r$.

Definition 4.2. For $u \in C(D)$, the set of L^2-selections of the multivalued map F is defined by

$$S^2_{F,u} = \{v \in L^2(D);\ v(x,t) \in F(x,t,u(x,t)),\ a.e.\,(x,t) \in D\}.$$

This set is not empty ([23, Lemma 3]).

Definition 4.3. Let $F : D \times \mathbb{R} \to 2^{\mathbb{R}}$ have nonempty compact values. The Nemitsky operator of F is the set-valued operator $\mathcal{F} : C(D) \to 2^{L^2(D)}$, defined by

$$\mathcal{F}(u) := \{w : D \to \mathbb{R} \text{ measurable};\ w(x,t) \in F(x,t,u(x,t)),\ a.e.\,(x,t) \in D\}.$$

Theorem 4.1. *(,[2732]). Let F be an L^2- Carathéodory multifunction. Then the Nemitsky operator $\mathcal{F}$ is weakly completely continuous and integrable on bounded sets.*

Using the properties of the Green's function we get the following results ([27]).

Lemma 4.1. *Assume the single valued map $g \in C(D)$. Then the operator $\gamma : Lip(D) \to C(D)$, defined by*

$$\gamma f(x,t) = g(x,t) + \int_0^t \int_\Omega G(x,t;y,s) f(y,s)dyds$$

is continuous.

Lemma 4.2. *Assume $F : D \times \mathbb{R} \to \mathbb{R}$ is an L^2- Carathéodory multifunction with nonempty, compact, convex values, with a Lipschitz continuous selection, and $g \in C(D)$. Then the operator $\gamma \circ \mathcal{F}$ is of usc type; i.e. is usc, completely continuous and has nonempty, compact, convex values.*

Theorem 4.2. *(Nonlinear alternative for multivalued maps) Let K be a convex subset of a Banach space E, $U \subseteq K$ be relatively open, and $0 \in U$. Suppose $\Lambda : \overline{U} \to K$ is an usc compact multivalued map with nonempty, compact, convex values. Then either*

(i) there is $u \in \overline{U}$ such that $u \in \Lambda u$; or

(ii) there is $u \in \partial U$ and a $\lambda \in (0, 1)$ such that $u \in \lambda\Lambda u$.

The above theorem can also be found in.[17]

Definition 4.4. Let (Z, d) be a metric space and let A, B be two nonempty subsets of Z. The Hausdorff distance between A and B is defined by

$$d_H(A, B) := \max\{\sup_{a \in A} d(a, B),\ \sup_{b \in B} d(A, b)\},$$

where $d(a, B) = \inf\{d(a, b); b \in B\}$ and $d(A, b) = \inf\{d(a, b); a \in A\}$. Then one can show that $(P_{cl,b}(Z),\ d_H)$ is a metric space. Here $P_{cl,b}(Z)$ denotes the collection of all closed bounded subsets of Z.

Definition 4.5. A multivalued operator $\Lambda : Z \to 2^Z$, with closed values is called

(i) $\delta-$Lipschitz if and only if there exists $\delta > 0$ such that $d_H\left(\Lambda(u), \Lambda(v)\right) \leq \delta\ d\left(u, v\right)$ for all $u, v \in Z$

(ii) a contraction if and only if it is $\delta-$Lipschitz with $\delta < 1$.

A subset $\Sigma \subset D \times \mathbb{R}$ is $\mathcal{L} \bigotimes \mathcal{B}$ measurable if Σ belongs to the $\sigma-$algebra generated by all sets of the form $\mathcal{D} \times \mathcal{J}$ where $\mathcal{D}$ is Lebesgue measurable in D and $\mathcal{J}$ is Borel measurable in $\mathbb{R}$.

Definition 4.6. Let (Z, d) be a separable metric space. We say that the multivalued operator $\Re : Z \to P\left(L^1(D; \mathbb{R})\right)$ has property (BC) if $\Re$ is lsc with nonempty, closed and decomposable values.

Remark 4.1. $\Re\left(z\right)$ is decomposable if for all $u, v \in \Re\left(z\right)$ and measurable $\Delta \subset D$, we have $u\chi_{\Delta} + v\chi_{D-\Delta} \in \Re\left(z\right)$, where χ_{Δ} is the characteristic function of the set Δ.

We state a selection theorem due to Bressan and Colombo ([4])

Theorem 4.3. *Let (Z, d) be a separable metric space and let $\Re : Z \to P\left(L^1(D; \mathbb{R})\right)$ has property (BC). Then $\Re$ has a continuous selection, i.e. there exists a single valued function $\rho : Z \to L^1(D; \mathbb{R})$ such that $\rho\left(z\right) \in \Re\left(z\right)$ for every $z \in Z$.*

5. Main results

We shall be interested in strong solutions of problem (1), (2), (3).

Definition 5.1. $u \in C\left(D\right)$ is called a strong solution of (1), (2), (3) if there exists a single-valued function $f \in Lip(D)$ such that $f(x, t) \in F(x, t, u(x, t))$ and (4), (5), (6) hold.

5.1. *Lipschitz multivalued right-hand sides*

Theorem 5.1. *Suppose that* $F : D \times \mathbb{R} \to 2^{\mathbb{R}}$ *is a multifunction with compact values and the following conditions are satisfied.*

(H0) There exists $f \in Lip(D)$ such that $f(x,t) \in F(x,t,u(x,t))$

Theorem 5.2.

(H1) There exists $\ell \in Lip(D)$ *such that*

$$d_H\left(F(x,t,u), F(x,t,z)\right) \le \ell\left(x,t\right)|u-z|, \quad a.e.\,(x,t) \in D, u,z \in \mathbb{R},$$
$$d_H(0, F(x,t,0) \le \ell\left(x,t\right) \quad a.e.\,(x,t) \in D,$$

(H2) $\max_{(x,t)\in\overline{D}}\{\int_0^t \int_\Omega G(x,t;\xi,\tau)\ell\left(\xi,\tau\right) d\xi d\tau < 1.$

(H3) $max_{x\in\Omega}\{\sum_{i=1}^m |\beta_i(x)| \int_\Omega G(x,t_i;y,0)dy\} < 1.$

Then Problem (1), (2), (3) has at least one solution.

Proof.

For $v \in C(\Omega)$ consider the problem (1), (2), (9) i.e.

$$\begin{aligned} D_t u + Lu &\in F(x,t,u), && (x,t) \in D,\\ u(x,t) &= 0, && (x,t) \in \Gamma,\\ u(x,0) &= v(x), && x \in \Omega. \end{aligned}$$

u is a solution of this problem if and only if $u \in X = C(D)$ is a fixed point of the multivalued operator

$$\Re : X \to 2^X,$$

defined by

$$\Re u = g_v + G\mathcal{F}(u) \tag{11}$$

where

$$g_v(x,t) = \int_\Omega G(x,t;y,0)\ v(y)dy, \quad (x,t) \in D, \tag{12}$$

and

$$G\mathcal{F}(u)(x,t) = \int_0^t \int_\Omega G(x,t;y,s)\ \mathcal{F}(u)\,(y,s)\,dyds, \quad (x,t) \in D. \tag{13}$$

Notice that $\Re$ is the sum of a single-valued operator g_v and a multivalued operator $G\mathcal{F}$, where $\mathcal{F}$ is the Nemitsky operator associated with the multifunction $F(x,t,u)$.

We show that $\Re u$ has closed and nonempty values for any $u \in X$. For, let $(z_n)_{n\in\mathbb{N}} \subset X$, $z_n \in \Re u$, $z_n \to z$ in X. Then, $z \in X$ and there exists $f_n \in Lip(D)$ such that

$$z_n(x,t) = g_v(x,t) + \int_0^t \int_\Omega G(x,t;y,s)\ f_n(y,s)dyds, \quad (x,t) \in D.$$

Since F has compact values, it follows that $(f_n)_{n\in\mathbb{N}}$, passing to subsequences if necessary, converges to some $h \in Lip(D)$. Hence

$$z(x,t) = g_v(x,t) + \int_0^t \int_\Omega G(x,t;y,s)\ h(y,s)dyds, \quad (x,t) \in D,$$

which shows that $z \in \Re u$, and also is nonempty.
Next, we show that $\Re$ is a contraction. For, let $u_1, u_2 \in X$ and consider $z_i \in \Re u_i$, $i = 1,2$. Then, there exist $h_i \in Lip(D)$, $i = 1,2$ such that

$$z_i(x,t) = g_v(x,t) + \int_0^t \int_\Omega G(x,t;y,s)\ h_i(y,s)dyds, \quad (x,t) \in D,\ i = 1,2.$$

Then

$$z_1(x,t) - z_2(x,t) = \int_0^t \int_\Omega G(x,t;y,s)\ [h_1(y,s) - h_2(y,s)]dyds, \quad (x,t) \in D.$$

(H1) yields

$$\begin{aligned} |z_1(x,t) - z_2(x,t)| &\le \int_0^t \int_\Omega G(x,t;y,s)\ell(y,s)\ |u_1(y,s) - u_2(x,t)|\,dyds \\ &\le \{\int_0^t \int_\Omega G(x,t;y,s)\ \ell(y,s)dyds\}\,\|u_1 - u_2\|_0 \\ &\le \max_{(x,t)\in\overline{D}} \{\int_0^t \int_\Omega G(x,t;y,s)\ell(y,s)\,dyds\}\,\|u_1 - u_2\|_0\,. \end{aligned}$$

This shows that

$$d_H(\Re u_1, \Re u_2) \le \delta \|u_1 - u_2\|_0\,,$$

where

$$\delta := \max_{(x,t)\in\overline{D}} \{\int_0^t \int_\Omega G(x,t;y,s)\ell(y,s)\,dyds\}.$$

It follows from (H2) that $\Re$ is a contraction. Theorem 11.1 in[11] implies that $\Re$ has at least one fixed point u_0. Then u_0 satisfies

$$u_0(x,t) \in \int_\Omega G(x,t;y,0)\ v(y)dy + \int_0^t \int_\Omega G(x,t;y,s)\ F(y,s,u_0(y,s))dyds$$

for $(x,t) \in D$. Lemma 3.3 shows that $u_0(x,t;v_0)$, where v_0 is the unique fixed point of the operator Υ, which exists by (H3), is a solution of problem (1), (2), (3).

5.2. *Upper semicontinuous right-hand sides*

Theorem 5.3. *Let $F : D \times \mathbb{R} \to \mathbb{R}$ be an L^2- Carathéodory multifunction with nonempty, compact, convex values. Assume that, in addition to (H0) and (H3), the following conditions are satisfied.*

(H4) There exist $q \in C(D)$ and $\Psi : [0,\infty) \to (0,\infty)$ continuous and nondecreasing such that $|F(x,t,u| \leq q(x,t)\Psi(|u|)$ for almost all $(x,t) \in D$ and $u \in \mathbb{R}$.

(H5) $sup_{\rho \in [0,\infty)} \dfrac{\rho}{|g_{v_0}|_0 + K_0 |q|_0 \Psi(\rho)} > 1.$

Then problem (1), (2), (3) has at least one solution.

Proof.

Recall that u is a solution of (1), (2), (3) if and only if u is a fixed point of the multivalued operator $\Re$ given by (11). Lemma 4.2 implies that $\Re$ is of upper semi-continuous type. It follows from condition (H3) and the properties of the Green's function that for $v_0 \in C(D)$, the unique fixed point of the operator Υ, the function g_{v_0} is well defined and continuous. Let $M_0 > 0$ satisfy

$$\frac{M_0}{|g_{v_0}|_0 + K_0 |q|_0 \Psi(M_0)} > 1.$$

This is possible because of (H5).
Consider $U := \{u \in X; |u|_0 < M_0\}$. Then U is relatively open in $K = X = C(\overline{D})$. We shall apply Theorem 4.2 to the operator $\Re$, and show that the second alternative does not hold. Let $u \in X$ be a solution of

$$u(x,t) \in \lambda(g_{v_0}(x,t) + \int_0^t \int_\Omega G(x,t;y,s)\ F(y,s,u(y,s))dyds), \quad (x,t) \in D, \tag{14}$$

with $\lambda \in (0,1)$. There exists a function $f \in Lip(D)$ such that $f(x,t) \in F(x,t,u(x,t))$.
From (12) we obtain

$$u(x,t) = \lambda(g_{v_0}(x,t) + \int_0^t \int_\Omega G(x,t;y,s)\ f(y,s)dyds), \quad (x,t) \in D. \tag{15}$$

Thus using (H4), we have for each $(x,t)\in D$

$$\begin{aligned}|u(x,t)| &\le |g_{v_0}(x,t)| + \int_0^t\int_\Omega G(x,t;y,s)\ q(y,s)\Psi\left(|u(y,s)|\right)dyds)\\ &\le |g_{v_0}|_0 + \int_0^t\int_\Omega G(x,t;y,s)\ q(y,s)dyds\Psi\left(|u|_0\right)\\ &\le |g_{v_0}|_0 + K_0\,|q|_0\,\Psi\left(|u|_0\right).\end{aligned}$$

Hence

$$|u|_0 \le |g_{v_0}|_0 + K_0\,|q|_0\,\Psi\left(|u|_0\right). \tag{16}$$

Suppose, now that there exists $u\in\partial U$ and $\lambda\in(0,1)$ such that $u\in\lambda\Re u$. Then u satisfies (13) and $|u|_0 = M_0$. It follows from (14) that

$$M_0 \le |g_{v_0}|_0 + K_0\,|q|_0\,\Psi\left(M_0\right).$$

This, obviously, contradicts the definition of M_0. Consequently, the first alternative in Theorem 4.2 holds; i.e. the multivalued operator $\Re$ has a fixed point u, such that $u(.,.;v_0)$ is a solution of (1), (2), (3).

5.3. *Lower semicontinuous right-hand sides*

Now, we present an existence result for nonconvex right-hand sides.

Theorem 5.4. *Assume, in addition to (H0), (H3), (H4) and (H5) that the following condition holds.*

(H6) $F : D\times\mathbb{R}\rightarrow P\left(\mathbb{R}\right)$ is a nonempty compact multivalued map such that

(a) $(x,t,u)\mapsto F(x,t,u)$ is $\mathcal{L}\bigotimes\mathcal{B}$ measurable,
(b) $u\mapsto F(x,t,u)$ is lsc for $a.e.(x,t)\in D$.
Then (1), (2), (3) has at least one solution.

Proof.

Conditions (H4), (H5) and (H6) imply that $\mathcal{F}$ has property (BC). Theorem 4.3 shows that there exists a continuous function $\eta : X\rightarrow L^1(D)$ such that $\eta\left(u\right)\in\mathcal{F}\left(u\right)$ for all $u\in X$.

For $v\in C\left(\Omega\right)$ consider the following single-valued problem

$$D_t u + Lu = \eta\left(u\right),\ \ (x,t)\in D, \tag{17}$$

$$u(x,t) = 0,\ \ (x,t)\in\Gamma, \tag{18}$$

$$u(x,0) = v\,(x)\,, \quad x \in \Omega. \tag{19}$$

We have seen that any solution $u \in X$ of (17), (18), (19) is a solution of the operator equation

$$\digamma u = g_v + G\eta(u),$$

where g_v is given by (12) and $G\eta(u)$ is given by

$$G\gamma(u)(x,t) = \int_0^t \int_\Omega G(x,t;y,s)\ \eta(u\,(y,s))dyds, \quad (x,t) \in D.$$

We can show that $\digamma$ is continuous and completely continuous.

Next, we proceed as in the proof Theorem 5.2 to show that any solution to $u = \lambda \digamma\,(u)$, for $\lambda \in (0,1)$ satisfies $|u|_0 \neq M_0$. Now, we apply Theorem 3.7 in[27] to show that $\digamma$ has a fixed point $u \in X$. Then $u\,(x,t;v_0)$, where v_0 is the fixed point of the operator Υ, is a solution of our problem.

5.4. *Method of lower and upper solutions*

In this part we consider the case of an L^2- Carathéodory multifunction of the form

$$F(x,t,u) = [\theta\,(x,t,u)\,, \kappa\,(x,t,u)]$$

where $\theta, \kappa : D \times \mathbb{R} \to \mathbb{R}$ are such that $\theta\,(\cdot,\cdot,u)\,, \kappa\,(\cdot,\cdot,u)$ are measurable and $\theta\,(x,t,\cdot)$ is lsc and $\kappa\,(x,t,\cdot)$ is usc. Thus F is an usc multifunction with nonempty, compact and convex values. $u \in X$ is a solution of (1), (2), (3) if there exists $f \in Lip\,(D)$ such that $\theta\,(x,t,u) \leq f(x,t) \leq \kappa\,(x,t,u)$ and (4), (5), (6) hold.

Definition 5.2. $z \in X$ is a lower solution of (1), (2), (3) if

$$D_t z + Lz \leq \theta\,(x,t,z)\,, \qquad (x,t) \in D,$$

$$u(x,t) \leq 0, \qquad (x,t) \in \Gamma,$$

$$z(x,0) + \sum_{i=1}^{m} \beta_i(x) z(x,t_i) \leq \phi\,(x)\,, \qquad x \in \Omega.$$

$Z \in X$ is an upper solution of (1), (2), (3) if $D_t Z + LZ \geq \kappa\,(x,t,z)\,,$ $(x,t) \in D$, and the last two inequalities are reversed when we substitute Z for z.

Theorem 5.5. *Assume that* $c(x,t) \geq c_0 > 0$ *on* D, $\beta_i(x) \leq 0$ *on* Ω *with* $-1 \leq \sum_{i=1}^{m} \beta_i(x) \leq 0$ *on* Ω, *and* $max_{x\in\Omega}\{\sum_{i=1}^{m} |\beta_i(x)| \int_\Omega G(x,t_i;y,0)dy\} < 1$. *Suppose that (1), (2), (3) has a lower solution* z *and an upper solution* Z *such that* $z(x,t) \leq Z(x,t)$. *Then the problem has at least one solution* $u \in [z,Z]$; *i.e.* $z(x,t) \leq u(x,t) \leq Z(x,t)$ *for every* $(x,t) \in D$.

Proof.

Let $\delta(u) = \max(z, \min(u,Z))$. Then $\delta : X \to [z,Z]$ is continuous and bounded with

$$|\delta(u)|_0 \leq \max(|z|_0, |Z|_0).$$

Consider the modified multifunction $F_1 : D \times \mathbb{R} \to P(\mathbb{R})$ defined by

$$F_1(x,t,u) = F(x,t,\delta(u)).$$

Then F_1 is an L^2- Carathéodory multifunction with nonempty, compact and convex values. Moreover, there exists $\omega_{F_1} \in L^2(D)$ such that

$$|F_1(x,t,u)| \leq \omega_{F_1}(x,t), \ \forall (x,t) \in D, \forall u \in \mathbb{R}.$$

It follows from Lemma 4.2 that the multivalued operator $\gamma \circ \mathcal{F}_1$ is of usc type. Here $\mathcal{F}_1$ is the Nemitsky operator of F_1 and γ is defined in Lemma 4.1.

For $v \in C(\Omega)$ we consider the modified problem

$$D_t u + Lu \in F_1(x,t,u), \ (x,t) \in D, \tag{22}$$

$$u(x,t) = 0, \quad (x,t) \in \Gamma, \tag{23}$$

$$u(x,0) = v(x), \ x \in \Omega, \tag{24}$$

whose solutions are given by

$$u(x,t) = g_v(x,t) + \int_0^t \int_\Omega G(x,t;y,s)\ f(y,s)\,dyds, \quad (x,t) \in D.$$

where g_v is given by (12) and $f(x,t) \in F_1(x,t,u(x,t))$, $(x,t) \in D$. Thus

$$|u|_0 \leq |g_v|_0 + |\varphi|_{L^2} |\omega_{F_1}|_{L^2} := r_0.$$

Let

$$Q := \{u \in X; |u|_0 \leq r_0\}.$$

Then Q is a nonempty, bounded, closed and convex subset of X. It follows from the properties of $\gamma\circ\mathcal{F}_1$ that $\overline{(\gamma \circ \mathcal{F}_1)(Q)}$ is compact and $(\gamma \circ \mathcal{F}_1)(Q) \subset Q$. By a theorem of Bohnenblust and Karlin (see Cor. 11.3 (e) in[11]) the operator $\gamma \circ \mathcal{F}_1$ has a fixed point u_1. Then $u_1(x,t;v_0)$, where v_0 is the fixed point of the operator Υ, is a solution of (22), (23), (24). To complete the proof we must show that $u_1 \in [z,Z]$.We show that $u_1(x,t) \geq z(x,t)$ for every $(x,t) \in D$. Suppose, on the contrary, that there exists $(\zeta,\tau) \in D$ such that $u_1(\zeta,\tau) < z(\zeta,\tau)$. Letting $w(x,t) = u_1(x,t) - z(x,t)$, we see that $w(\zeta,\tau) < 0$. w achieves a negative minimum at some point $(\zeta_0,\eta_0) \in \overline{D}$. By the strong maximum principle (see[16]) we have either $(\zeta_0,\eta_0) \in \Gamma$ or $(\zeta_0,\eta_0) = (\zeta_0,0)$ with $\zeta_0 \in \Omega$. Since $w(x,t) \geq 0$ for $(x,t) \in \Gamma$, we see that

$$\min_{\overline{D}} w(x,t) = w(\zeta_0,0) < 0.$$

We now proceed as in the proof of Lemma 3.1 to complete the proof. Similarly, we show that $u_1(x,t) \leq Z(x,t)$ for every $(x,t) \in D$. Now, for $u \in [z,Z]$ we have that $\delta(u) = u$. Hence $F_1(x,t,u)$ and $F(x,t,u)$ coincide. Consequently, u_1 is a solution of the original problem.

Acknowledgements: The first draft of this paper was written while the author was visiting the Division of Applied Mathematics at Brown University through a Grant from The Arab Fund For Economic And Social Development, Kuwait. He is grateful to both Institutions and Professor John Mallet-Paret for making the visit possible. Also, he wishes to thank King Fahd University of Petroleum and Minerals for its constant support.

References

1. J. P. Aubin and A. Cellina, *Differential Inclusions*, Springer Verlag, Berlin, New york, 1984.
2. J. P. Aubin and H. Frankowska, *Set-Valued Analysis*, Birkhauser, Boston, 1990.
3. K. Balachandran and K. Uchiyama, Existence of solutions of nonlinear integrodifferential equations of Sobolev type with nonlocal conditions in Banach spaces, Proc. Indian Acad. Sci. (Math. Sci.), 110 (2000), 225-232
4. A. Bressan and G. Colombo, Extensions and selections of maps with decomposable values, Studia Math. 90 (1988), 69-86
5. L. Byszewski, Theorems about the existence and uniqueness of solutions of a semilinear evolution nonlocal Cauchy problem, J. Math. Anal. Appl. 162 (1991), 494-505.
6. T. Cardinali, A. Fiacca and N. S. Papageorgiou, *Extremal solutions for nonlinear parabolic problems with discontinuities*, Mh. Math. 124 (1997), 119-131.
7. S. Carl and S. Heikkilä, *On a parabolic boundary value problem with discontinuous nonlinearity*, Nonl. Anal. 15 (1990), 1091-1095.
8. S. Carl, C. Grossmann and C. V. Pao, *Existence and monotone iterations for parabolic differential inclusions*, Comm. Appl. Nonl. Anal. 3 (1996), 1-24.
9. J. Chabrowski, *On nonlocal problems for parabolic equations*, Nagoya Math. J. 93 (1984), 109-131.

10. H. Deguchi, *On weak solutions of parabolic initial value problems with discontinuous nonlinearities*, Nonl. Anal. 63 (2005), e1107-e1117.
11. K. Deimling, *Multivalued Differential Equations*, W. de Gruyter, Berlin, 1992.
12. K. Deng, Exponential decay of solutions of semilinear parabolic equations with nonlocal initial conditions, J.Math. Anal. Appl. 179 (1993), 630-637.
13. E. Feireisl, *A note on uniqueness for parabolic problems with discontinuous nonlinearities*, Nonl. Anal. 16 (1991), 1053-1056.
14. E. Feireisl and J. Norbury, *Some existence, uniqueness and nonuniqueness theorems for solutions of parabolic equations with discontinuous nonlinearities*, Proc. Royal Soc. Edinburgh 119 A (1991), 1-17.
15. B. A. Fleishman and T. J. Mahar, A step function model in chemical reactor theory: Multiplicity and stability of solutions, Nonl. Anal. 5 (1981), 645-654.
16. A. Friedman, *Partial Differential Equations of Parabolic Type*, Prentice-Hall, Englewood Cliffs, New Jersey, 1964.
17. M. Frigon, *Application de la théorie de la transversalité topologique à des problèmes non linéaires pour des équations differentielles ordinaires*, Dissertationes Math. 296 (1990), 1-79.
18. J. Hofbauer and P. L. Simon, *An existence theorem for parabolic equations on* $\mathbb{R}^N$ *with discontinuous nonlinearity*, Electron. J. Qual. Theory Diff. Eqns. 2001, No. 8, pp.1-9
19. S. Hu and N. S. Papageorgiou, *Handbook of Multivalued Analysis, Vol. I: Theory*, Kluwer, Dordrecht, The Netherlands, 2000.
20. D. Jackson, *Existence and uniqueness of solutions to semilinear nonlocal parabolic equations*, J. Math. Anal. Appl. 172 (1993), 256-265.
21. A. A. Kerefov, *Nonlocal boundary value problems for parabolic equations*, Differential Equations 15 (1979), 52-54.
22. O. A. Ladyzhenskaya, V. A. Solonnikov and N. N. Uraltseva, *Linear and Quasilinear Equations of Parabolic Type*, Nauka, Moscow, 1967 [English translation: Amer. Math. Soc., Providence, R.I. 1968.]
23. A. Lasota and Z. Opial, *An application of the Kakutani-Ky Fan theorem in the theory of ordinary differential equations*, Bull. Acad. Pol. Sci. Ser. Sci. Math. Astronom. Phys. 13 (1965), 781-786.
24. G. M. Lieberman, *Second Order Parabolic Differential Equations*, World Scientific, River Edge, New York, 1996.
25. G. M. Lieberman, *Nonlocal problems for quasilinear parabolic equations*, in "Nonlinear Problems in Mathematical Physics and related Topics I", Kluwer 2002, 247-270.
26. Y. Lin, *Analytical and numerical solutions for a class of nonlocal nonlinear parabolic differential equations*, SIAM J. Math. Anal. 25 (1994), 1577-1594.
27. D. O'Regan, *Integral inclusions of upper semi-continuous or lower semi-continuous type*, Proc. Amer. Math. Soc. 124 (1996), 2391-2399
28. C. V. Pao, *Nonlinear Parabolic and Elliptic Equations*, Plenum Press, New York, 1992.
29. N. S. Papageorgiou, *On the existence of solutions for nonlinear parabolic problems with nonmonotone discontinuities*, J. Math. Anal. Appl. 205 (1997), 434-453.

30. V. N. Pavlenko and O. V. Ul'yanova, *Method of upper and lower solutions for parabolic-type equations with discontinuous nonlinearities*, Differential Equations 38 (2002), 520-527.
31. R. Pisani, *Problemi al contorno per operatori parabolici con non linearita discontinua*, Rend. Ist. Mat. Univ. Trieste 14 (1982), 85-98.
32. T. Pruszko, *Topological degree methods in multivalued boundary value problems*, Nonl. Anal. 5 (1981), 953-973
33. J. Rauch, *Discontinuous semilinear differential equations and multiple valued maps*, Proc. Amer. Math. Soc. 64 (1977), 277-282.
34. A. A. Samarskii, *Some problems in differential equation theory*, Differential Equations 16 (1980), 1221-1228.
35. P. N. Vabishchevich, *Nonlocal parabolic problems and the inverse heat-conduction problem*, Differential Equations 17 (1981), 761-765.

Periodic solutions of nonlinear parabolic equations with measure data and polynomial growth in $|\nabla u|$

Abderrahmane El Hachimi* and Abdelilah Lamrani Alaoui†
UFR Mathématiques Appliquées et Industrielles, Faculté des Sciences
B.P 20, El Jadida, Morocco
E-mails: * *elhachimi@ucd.ac.ma,* † *lamranii@gmail.com*

In this paper we are interested in the problem:

$$(\mathcal{P})\begin{cases} \dfrac{\partial u}{\partial t} - \Delta_p u = f(x,t,u,\nabla u) + \lambda\mu & \text{in } Q_T \\ \qquad u = 0 & \text{on } \Sigma_T \\ \quad u(0) = u(T) & \text{in } \Omega \end{cases}$$

where f is a Carathéodory function with polynomial growth in $|\nabla u|$, λ is a real number, μ a bounded Radon measure and $p \geq 2$.

Keywords: Periodic solution; Nonlinear parabolic equation.

1. Introduction

Periodic solutions of parabolic problems were studied by many authors and in many context particulary when the elliptic operator is linear (see[10,12] and the references therein). In the case of measure datum the unique paper we found is[1] where $(\mathcal{P})$ with $p = 2$ was studied. In that work the authors firstly give necessary conditions on the datum for existence, they show that a size and regularity information is consequence of existence in some cases. Secondly they give existence in the case of special measure, that is an L^1 function.

In the present work we generalize in different directions the results in,[1] firstly the elliptic operator is no more linear it is the so called degenerate $p-$Laplacian, that is

$$-\Delta_p u = -\text{div}(|\nabla u|^{p-2}\nabla u), \;\; p \geq 2.$$

Secondly the existence in the natural growth case is given for sufficiently regular measures (in some capacity sense) but not necessarily in $L^1(Q_T)$. The notion of capacity considered here is the one adapted to the parabolic

$p-$Laplacian operator due to Droniou et al.[8]
Let us mention that the absence of information about the initial value of the solution causes some technical difficulties.
Throughout the paper we adopt the following notations:

- Ω an open bounded subset of $I\!R^N$, $\partial\Omega$ its boundary, $T > 0$, $\mathcal{Q}_T = \Omega \times (0,T)$ and $\Sigma_T = \partial\Omega \times (0,T)$,
- $\mathcal{V}_0 = L^p(0,T; W_0^{1,p}(\Omega))$ and $\mathcal{V}_0^{'} = L^{p'}(0,T; W^{-1,p'}(\Omega))$ its dual where $p' = \dfrac{p}{p-1}$,
- $W = \left\{ u \in \mathcal{V}_0 \text{ such that } \dfrac{\partial u}{\partial t} \in \mathcal{V}_0^{'} + L^1(\mathcal{Q}_T) \right\}$
- $W_r^{1,1}(\mathcal{Q}_T)$ is the set of all measurable functions u such that
$$\int_{\mathcal{Q}_T} \left(|u|^r + |Du|^r + |\frac{\partial u}{\partial t}|^r \right) dxdt < +\infty$$
- $\mathcal{C}_b(\Omega)$ the space of bounded and continuous function on Ω,
- $\mathcal{M}_b(\mathcal{Q}_T)$ the space of bounded Radon measures,
- $\mathcal{M}_0(\mathcal{Q}_T)$ the subspace of $\mathcal{M}_b(\mathcal{Q}_T)$ which don't charge sets of parabolic capacity zero (see section 3),
- $\ll .,. \gg$ is the duality between ${\mathcal{V}'}_0$ and $\mathcal{V}_0$.
- $< .,. >$ is the duality between $W^{-1,p'}(\Omega)$ and $W_0^{1,p}(\Omega)$,
- $|.|$ is the Lebesgue measure of a borelian set, the euclidian norm of a vector or the absolute value of a real number,
- $\|.\|_X$ the norm of an element of a Banach space X. If $X = L^p$ with $p \in [1, +\infty]$ it will be noted $\|.\|_p$,
- C is a generic positive constant where the dependence on parameters will be indicated only if needed.

The following hypothesis is common to all sections.

$$f \text{ is a Carathéodory function:}$$

$$\text{measurable in } (t,x) \text{for all} (r,\xi) \in I\!R^{N+1} \tag{1}$$

$$\text{and continuous in } (r,\xi) \text{ for almost every } (t,x) \in \mathcal{Q}_T.$$

The paper is organized as follows. Section 2 is devoted to necessary conditions of existence, regularity of the measure and nonexistence result are established. Subsequently, we present existence result in the case of natural growth and some regular measures datum.

The paper is organized as follows. Section 2 is devoted to necessary conditions of existence, regularity of the measure and nonexistence result

are established. Subsequently, we present existence result in the case of natural growth and some regular measures datum.

2. Necessary conditions for existence

In this section we are interested in the following problem:
for $\lambda \in I\!R$ find u such that:

$$\begin{aligned} & u \in L^p(0,T,W_0^{1,p}(\Omega)) \cap \mathcal{C}(]0,T[,L^1(\Omega)) \\ & f(x,t,u,\nabla u) \in L^1_{loc}(\mathcal{Q}_T), \\ & \frac{\partial u}{\partial t} - \Delta_p u \geq f(x,t,u,\nabla u) + \lambda\mu \text{ in } \mathcal{D}^{'}(\mathcal{Q}_T) \\ & u(0) = u(T) \text{ in } \mathcal{M}_b\ (\Omega), \end{aligned} \tag{2}$$

where

$$\mu \text{ is nonnegative bounded Radon measure.} \tag{3}$$

The periodicity condition $u(0) = u(T)$ in $\mathcal{M}_b(\Omega)$ is understood in the following sense

$$\lim_{t\to 0^+} \int_\Omega (u(x,t) - u(x,T-t))\varphi dx = 0$$

for all $\varphi \in \mathcal{C}_b(\Omega)$.

2.1. *Nonexistence in supercritical case ($q > 2(p-1)$)*

In this subsection we prove that solutions to (2) does not exists for all values of λ if the growth of f in $|\nabla u|$ is sufficiently large.
We assume that there exists

$$]\varepsilon,\sigma[\text{ open in }]0,T[, \tag{4}$$

$$q > 2(p-1) \tag{5}$$

and

$$\text{a constant } C_0 > 0 \text{ such that} f(x,t,s,\xi) \geq C_0|\xi|^q$$

$$\text{for almost every } (t,x) \in]\varepsilon,\sigma[\ \times\Omega := \mathcal{Q}_{\varepsilon,\sigma} \text{and for all } (s,\xi) \in I\!R^{N+1} \tag{6}$$

and finally

$$\mu(]\varepsilon,\sigma[\times\Omega) > 0. \tag{7}$$

The main result of this subsection is the following:

Theorem 2.1. *Assume that (1), (3) and (4)-(7) hold, then there exists finite λ^* such that (2) does not have any solution for $\lambda > \lambda^*$.*

Remark 2.1. The condition (5) is reduced to $q > 2$ in the case $p = 2$ which is the superquadratic case treated in[1] and.[2]

Now we prove Theorem 2.1.

Proof. We follow here the proofs of[1] and[2] with some changes due to the nonlinear feature of the problem (2).
If u is a solution of (2), then by (6) we have

$$\frac{\partial u}{\partial t} - \Delta_p u \geq C_0|\nabla u|^q + \lambda\mu \text{ in } \mathcal{D}'(\mathcal{Q}_{\varepsilon,\sigma}) \tag{8}$$

let $\phi \in \mathcal{C}_0^\infty(]0,T[\times\Omega)$ such that $\phi \geq 0$ and $\phi(\varepsilon) = \phi(\sigma) = 0$.
Multiply (2) by ϕ and integrate from ε to σ we obtain

$$\lambda\int_{\mathcal{Q}_{\varepsilon,\sigma}} \phi d\mu \leq \int_{\mathcal{Q}_{\varepsilon,\sigma}} |\nabla u|^{p-1}|\nabla\phi| - C_0\int_{\mathcal{Q}_{\varepsilon,\sigma}} |\nabla u|^q\phi - \int_{\mathcal{Q}_{\varepsilon,\sigma}} u\partial_t\phi. \tag{9}$$

This inequality can be extended to $\phi \in \mathcal{C}^1([0,T];L^\infty(\Omega)) \cap L^\infty(0,T;W_0^{1,\infty}(\Omega))$, $\phi \geq 0$ and $\phi(\varepsilon) = \phi(\sigma) = 0$.
But $\phi_t = -\Delta G(\phi_t) = -\Delta(G\phi)_t$ where G is the Green Kernel on Ω.
Then from (9) and using twice Young inequality we obtain

$$\lambda\int_{\mathcal{Q}_{\varepsilon,\sigma}} \phi d\mu \leq C\left(\int_{\mathcal{Q}_{\varepsilon,\sigma}} \frac{|\nabla\phi|^{\frac{q}{q-p+1}}}{\phi^{\frac{p-1}{q-p+1}}} + \int_{\mathcal{Q}_{\varepsilon,\sigma}} \frac{|\nabla(G\phi)_t|^{q'}}{\phi^{\frac{1}{q-1}}}\right). \tag{10}$$

For $\phi = (\sigma-t)^q(t-\varepsilon)^q\Phi$, where Φ is a solution of the eigenvalue problem

$$-\Delta\phi = \lambda_1\phi \text{ in } \Omega, \phi > 0 \text{ in } \Omega, \phi = 0 \text{ on } \partial\Omega. \tag{11}$$

Where λ_1 is the first eigenvalue of $-\Delta$ in Ω.
We have

$$(G\phi)_t = \frac{q}{\lambda_1}\Phi(t-\varepsilon)^{q-1}(\sigma-t)^{q-1}(\sigma+\varepsilon-2t)$$

Using (5) and the fact that $\phi \in W_0^{1,\infty}(\Omega)$ and $\dfrac{1}{\phi^\alpha} \in L^1(\Omega)$ for all $\alpha < 1$. the two terms of the right hand side of (10) are finite.
Finally using (3) the existence of a finite λ^* such that: existence for (2) implies $\lambda \leq \lambda^*$. □

2.2. *Existence implies regularity of the measure*

We recall first that a compact set K in $\mathcal{Q}_T$ is of $W_r^{1,1}(\mathcal{Q}_T)$-capacity zero where $r \geq 1$, if there exists a sequence of $\mathcal{C}_0^\infty(\mathcal{Q}_T)$ functions $(\phi_n)_n$ such that :

$$0 \leq \phi_n \leq 1 \text{ a.e in } \mathcal{Q}_T, \phi_n \equiv 1 \text{ on } K, \phi_n \to 0 \text{ in } W_r^{1,1}(\mathcal{Q}_T) \text{ and a.e. in } \mathcal{Q}_T. \tag{12}$$

The main result of this subsection concerns the problem (2) when:

$$\text{There exists } q > p-1, \tag{13}$$

and

$$\text{there exists } C_1 \text{ and } C_2 \text{ such that } f(x,t,s,\xi) \geq C_1|\xi|^q - C_2 \text{ in } \mathcal{Q}_T \times I\!R^{N+1}. \tag{14}$$

Theorem 2.2. *Assume that (1), (3) hold, and that f satisfies (14) and (13). If (2) has a solution then μ does not charge the sets of $W_r^{1,1}(\mathcal{Q}_T)$-capacity zero where $r = \dfrac{q}{q-p+1}$.*

Remark 2.2. The above statement means that:

$$\big(K \text{ compact}, W_r^{1,1}(\mathcal{Q}_T)\text{-capacity}(K) = 0\big) \Rightarrow \mu(K) = 0.$$

This result extend to the $p-$Laplacian the one of[1] in the Laplacian case $(p = 2)$ where r was $\dfrac{q}{q-1}$.
In the case $q > 2(p-1)$ both size and regularity conditions are necessary for existence.

Proof. From (2) and (14) we have:

$$\frac{\partial u}{\partial t} - \Delta_p u \geq C_1|\nabla u|^q - C_2 + \lambda\mu \text{ in } \mathcal{D}^{'}(\mathcal{Q}_T). \tag{15}$$

Let K a compact set of $W_r^{1,1}(\mathcal{Q}_T)$-capacity zero and $(\phi_n)_n \subset \mathcal{C}_0^\infty(\mathcal{Q}_T)$ such that (12) is satisfied.
Multiplying (15) by $\psi_n = {\phi_n}^r$ (where r is given in Theorem 2.2) we obtain

$$\lambda\mu(K) + C_1\int_{\mathcal{Q}_T}|\nabla u|^q\psi_n \leq -\int_{\mathcal{Q}_T} u\frac{\partial\psi_n}{\partial t} + \int_{\mathcal{Q}_T}|\nabla u|^{p-1}|\nabla\psi_n| + C_2\int_{\mathcal{Q}_T}\psi_n. \tag{16}$$

By Young inequality we have:

$$\int_{Q_T} |\nabla u|^{p-1}|\nabla \psi_n| \le \varepsilon \int_{Q_T} |\nabla u|^q \psi_n + C(\varepsilon) \int_{Q_T} |\nabla \phi_n|^r. \tag{17}$$

For ε sufficiently small and with (17) into (16) we obtain

$$\lambda\mu(K) \le -r \int_{Q_T} u.{\phi_n}^{r-1}\partial_t \phi_n + C \int_{Q_T} |\nabla \phi_n|^r + C_2 \int_{Q_T} \phi_n^r. \tag{18}$$

Passing to the limit using Vitali theorem and (12) we have $\mu(K) = 0$ □

3. Existence in the case of natural growth and regular measure data

In this section we focus our attention on the problem (**Proof**.) where f takes the form

$$f(x, t, s, \xi) = -g(s)|\xi|^p$$

with

$$g \text{ a continuous fonction on } I\!R \text{ such that } g \in L^1_+(I\!R). \tag{19}$$

That is :

$$\frac{\partial u}{\partial t} - \Delta_p u + g(u)|\nabla u|^p = \mu \text{ in } Q_T u = 0 \text{ on } \Sigma_T u(0) = u(T) \text{ in } \Omega. \tag{20}$$

Remark 3.1. Because $p \ge 2$ we have $p \le 2(p-1)$ then no size condition on the measure is needed in this section and the parameter λ is dropped.

The lack of initial data by the periodicity condition leads to some difficulties which we overcome by considering the following problem.

$$\begin{cases} u \ge g_2 & \text{a.e. in } Q_T \\ \dfrac{\partial u}{\partial t} - \Delta_p u + g(u)|\nabla u|^p = \mu & \text{in } Q_T \\ u = 0 & \text{on } \Sigma_T \\ u(0) = u(T) & \text{in } \Omega \end{cases} \tag{21}$$

where μ is a regular measure in the sense $\mu \in \mathcal{L}(Q_T) = \mathcal{V}_0' + L^1(Q_T) + \partial_t(\mathcal{V}_0 \cap L^\infty(Q_T))$ that is

$$\mu = g_0 + g_1 + \partial_t g_2$$

where $(g_0, g_1, g_2) \in \mathcal{V}_0' \times L^1(Q_T) \times (\mathcal{V}_0 \cap L^\infty(Q_T))$.
Before giving the meaning of solutions for the problem (21) let us review some well known results we use here and explain the reason for that choice of measures.

3.1. *Some known results*

We begin with a decomposition for measures which do not charge sets of parabolic capacity zero.
First, recall that, if $U \subset \mathcal{Q}_T$ is an open set, parabolic capacity of U is defined as follows:

$$Cap_p(U) = \inf\{\|u\|_W : u \in W, u \geq \chi_u \text{a.e in } \mathcal{Q}_T\}$$

Then for any borelian set $B \subset \mathcal{Q}_T$

$$Cap_p(B) = \inf\{Cap_p(U), U \text{ open subset of } \mathcal{Q}_T, B \subset U\}.$$

For more properties and details on the parabolic capacity see.[8]

Proposition 3.1. *(theorem 2.28[8]) If $\mu \in \mathcal{M}_0(\mathcal{Q}_T)$, the subspace of measures in $\mathcal{M}_b(\mathcal{Q}_T)$ which don't charge sets of parabolic capacity zero.*
Then there exists $(g_0, g_1, g_2) \in \mathcal{V}_0' \times L^1(\mathcal{Q}_T) \times \mathcal{V}_0$ such that :

$$\int_{\mathcal{Q}_T} \phi d\mu = \int_0^T < g_0, \phi > dt + \int_{\mathcal{Q}_T} g_1 \phi dxdt - \int_0^T < (\phi)_t, g_2 > dt \ \ \forall \phi \in \mathcal{D}(\mathcal{Q}_T).$$

We now give a trace and integration by part results which make it possible to define and treat periodicity in (21).

Proposition 3.2. *(theorem 1.1[13] and Lemma 2[15]) We have:*

$$W \hookrightarrow \mathcal{C}([0,T]; L^1(\Omega)),$$

and for $v \in W$, ϕ Lipschitz bounded real function from $I\!R$ to $I\!R$, with $\phi(0) = 0$ we have for all $t \in [0,T]$

$$\int_0^t < \frac{\partial v(\sigma)}{\partial t}, \phi(v(\sigma)) > d\sigma = \int_\Omega dx \int_0^{v(x,t)} \phi(\sigma) d\sigma - \int_\Omega dx \int_0^{v(x,0)} \phi(\sigma) d\sigma$$

Moreover if $v(0) = v(T)$ in $L^1(\Omega)$ we have $\ll \frac{\partial}{\partial vt}, \phi(v) \gg = 0$.

Now we are ready to give the main result of this section.

3.2. *Definition and main result*

(,[45] and[7]) Let μ be a measure in $\mathcal{L}(\mathcal{Q}_T)$ with a decomposition (g_0, g_1, g_2), a measurable function $u \geq g_2$ is said to be renormalized solution of (21) if

$$T_k(u) \in \mathcal{V}_0, \tag{22}$$

$$T_k(u - g_2) \in \mathcal{V}_0, \tag{23}$$

for every $k > 0$, and

$$\lim_{n\to+\infty} \int_{\{n\le u-g_2\le n+1\}} |\nabla u|^p dxdt = 0, \tag{24}$$

and, for every $S \in W^{2,\infty}(I\!R)$ such that S' has compact support in $I\!R^+$, $S(0) = 0$,

$$\begin{aligned} &\partial_t S(u-g_2) - \operatorname{div}(S'(u-g_2)|\nabla u|^{p-2}\nabla u) \\ &+S''(u-g_2)|\nabla u|^{p-2}\nabla u.\nabla(u-g_2) \\ &+g(u)|\nabla u|^p S'(u-g_2) \end{aligned}$$

$$= S'(u-g_2)g_1 + G_0 S''(u-g_2)\nabla(u-g_2) - \operatorname{div}(G_0 S'(u-g_2)) \tag{25}$$

in the sense of distributions, with $g_0 = \operatorname{div}(G_0)$,

$$S(u-g_2)(0) = S(u-g_2)(T). \tag{26}$$

We state our main result

Theorem 3.1. *Let μ a measure in $\mathcal{L}(\mathcal{Q}_T)$ and assume that (19) holds, then (21) has a solution in the sense of definition above.*

The proof of this result is divided into three classical steps:
- Regularization.
- Existence of approximate solutions.
- A priori estimates and passing to the limit.

3.3. *Regularization and existence of approximate solutions*

Let (g_0, g_1, g_2) be a decomposition of μ, we regularize (21) by the sequence of problems

$$\begin{cases} \frac{\partial u_n}{\partial t} - \Delta_p u_n + g(u_n)|\nabla u_n|^p - nT_n(u_n - g_2^+)^- = \mu_n \text{ in } \mathcal{Q}_T \\ u_n = 0 \text{ on } \Sigma_T \\ u_n(0) = u_n(T) \text{ in } \Omega \end{cases} \tag{27}$$

where,

$$\mu_n = g_0^n + g_1^n + (g_2^n)_t, \text{with } g_i^n \in \mathcal{C}_c^\infty(\mathcal{Q}_T),\ i = 1,2,3 \text{ and } \|\mu_n\|_{L^1(\mathcal{Q}_T)} \le C, \tag{28}$$

and

$$g_0^n \to g_0 \text{ in } \mathcal{V}_0', \tag{29}$$

$$g_1^n \to g_1 \text{ in } L^1(\mathcal{Q}_T), \tag{30}$$

$$g_2^n \to g_2 \text{ in } \mathcal{V}_0 \text{ and a.e in} \mathcal{Q}_T, \text{ with } \|g_2^n\|_\infty \le \|g_2\|_\infty. \tag{31}$$

Let us prove the existence of approximate solutions, to do this we show for fixed $n \in I\!N$ existence of ordered lower and upper solutions to (27).
We begin by showing existence of an upper solution.
Let $\tilde{\beta_n} \in W_0^{1,p}(\Omega) \cap \mathcal{C}^1(\overline{\Omega})$ be the solution of the Torsion problem:

$$-\Delta_p u = \|\mu_n\|_\infty + n^2 \text{ in } \Omega, u = 0 \text{ on } \partial\Omega, \tag{32}$$

and $\alpha_n \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, see,[6] the solution of the problem

$$-\Delta_p u + g(u)|\nabla u|^p = -\|\mu_n\|_\infty \text{ in } \Omega, u = 0 \text{ on } \partial\Omega, \tag{33}$$

By taking $\beta_n = \tilde{\beta_n} + \beta$ where β is a large real number such that $\beta_n \ge \max(\alpha_n, g_2)$ almost everywhere in $\mathcal{Q}_T$. □

3.4. *A priori estimates and passage to the limit*

We have constructed a pair of ordered lower and upper solutions to (27), now we use[9] to derive the existence of a solution $u_n \in \mathcal{Q}_0 \cap L^\infty(\mathcal{Q}_T)$, $u_n \in \mathcal{Q}([0,T]; L^s(\Omega))$ for all $s \ge 1$, and $\alpha_n \le u_n \le \beta_n$ a.e. in $\mathcal{Q}_T$
We collect some a priori estimates, we begin by

Lemma 3.1. *$(nT_n(u_n - g_2^n)^-)_n$ is bounded in $L^1(\mathcal{Q}_T)$.*

Proof. Fix $k > \|g_2\|_\infty$, let $h > 0$ and consider $\varphi_n = T_h(u_n - T_k(u_n))\exp(-G(u_n))$ where $G(s) = \int_0^s g(\sigma)d\sigma$ for all s in $I\!R$ as a test function in (27), we obtain

$$\int_{\mathcal{Q}_T} |\nabla u_n|^p exp(-G(u_n))\chi_{\{k<|u_n|<h+k\}} - n\int_{\mathcal{Q}_T} T_n(u_n - g_2^n)^- \varphi_n = \int_{\mathcal{Q}_T} \mu_n\varphi_n$$

but

$$\exp(-\|g\|_{L^1(I\!R)}) \le \exp(-G(s)) \le \exp(\|g\|_{L^1(I\!R)}) \text{ for all } s \in I\!R \tag{34}$$

then (28) gives

$$-n\int_{\mathcal{Q}_T} T_n(u_n - g_2^n)^- T_h(u_n - T_k(u_n))exp(-G(u_n)) \le Ch,$$

the choice of k gives by dividing on h and passing to the limit as h tends to 0

$$n\int_{\mathcal{Q}_T} T_n(u_n - g_2^n)^- \le C.$$

The proof is then complete. □

Proposition 3.3. *For fixed $k > 0$, let u_n be solution to (27), then for all $n \in I\!N$*

$$\int_{\mathcal{Q}_T} |\nabla T_k(u_n)|^p \leq Ck, \tag{35}$$

$$\int_{\mathcal{Q}_T} |\nabla T_k(u_n - g_2^n)|^p \leq C(k+1), \tag{36}$$

$$\lim_{h\to+\infty} \sup_{n\in I\!N} |\{|u_n - g_2^n| \geq h\}| = 0. \tag{37}$$

and

$$\lim_{h\to+\infty} \left(\sup_n \int_{h\leq u_n - g_2^n \leq h+k} |\nabla u_n|^p dxdt \right) = 0. \tag{38}$$

Moreover there exists a measurable function $u : \mathcal{Q}_T \to I\!R$ such that $u \geq g_2$ a.e. in $\mathcal{Q}_T$ and $T_k(u), T_k(u - g_2)$ belong to $\mathcal{V}_0$ and, up to a subsequence

$$u_n \to u \text{ a.e. in } \mathcal{Q}_T, T_k(u_n - g_2^n) \rightharpoonup T_k(u - g_2) \text{ in } \mathcal{V}_0 \text{ and a.e. in } \mathcal{Q}_T. \tag{39}$$

Finally

$$\lim_{h\to+\infty} \int_{h\leq u - g_2 \leq h+k} |\nabla u|^p dxdt = 0. \tag{40}$$

Proof. In all that follows, we set $v_n = u_n - g_2^n$ and $v = u - g_2$. By taking $T_k(u_n)\exp(-G(u_n))$ as test function in (27) and using the periodicity of u_n, (34) we obtain

$$\begin{array}{l} C\int_{\mathcal{Q}_T} |\nabla T_k(u_n)|^p \leq \int_{\mathcal{Q}_T} \mu_n T_k(u_n)\exp(-G(u_n)) \\ +n\int_{\mathcal{Q}_T} T_n(u_n - g_2^n)^- T_k(u_n)\exp(-G(u_n)). \end{array} \tag{41}$$

Then (28) and Lemma 3.1 gives (35).
For (36) we have

$$\begin{aligned} \int_{\mathcal{Q}_T} |\nabla T_k(v_n)|^p &= \int_{\mathcal{Q}_T} |\nabla T_k\left(T_{M_k}(u_n) - g_2^n\right)|^p \\ &= \int_{\mathcal{Q}_T} |\nabla\left(T_{M_k}(u_n) - g_2^n\right)|^p \chi_{\{|v_n|\leq k\}} \\ &\leq C\int_{\mathcal{Q}_T} \left(|\nabla T_{M_k}(u_n)|^p + |\nabla g_2^n|^p\right) \chi_{\{|v_n|\leq k\}} \\ &\leq C(M_k+1) \leq C(k+1) \end{aligned}$$

where $M_k = \|g_2\|_\infty + k$. Now we prove (37), to do this we follow.[3]
Fix $h > 0$, let $\varepsilon \in]0, h[$ we have :

$$|\{|v_n| \geq \varepsilon\}| = |\{|T_h(v_n)| \geq \varepsilon\}| \leq \int_{\mathcal{Q}_T} \frac{|T_h(v_n)|^p}{\varepsilon^p} = \frac{\|T_h(v_n)\|_p^p}{\varepsilon^p}.$$

Then

$$|\{|v_n| \geq \varepsilon\}| \leq C\frac{\|\nabla T_h(v_n)\|_p^p}{\varepsilon^p} \leq C\frac{h+1}{\varepsilon^p},$$

taking for example $\varepsilon = \dfrac{h}{2}$ we obtain:

$$|\left\{|v_n| \geq \frac{h}{2}\right\}| \leq C\frac{h+1}{h^p}$$

but $p \geq 2$, then (37) is proved.
Now we show the existence of a measurable function u such that

$$u_n \to u \text{ a.e in } \mathcal{Q}_T, \text{and } T_k(u), T_k(u - g_2) \in \mathcal{V}_0 \text{ for all } k \in I\!N. \tag{42}$$

We adapt to our case the method used in[8] (see also[3]).
Given $\varepsilon > 0$, fix (by the help of (37)) $k \in I\!N$ such that

$$|\left\{|v_n| > \frac{k}{2}\right\}| < \varepsilon, \text{ independently of } n. \tag{43}$$

Let $\mathcal{T}_k \in \mathcal{C}^2(I\!R)$ nondecreasing function such that $\mathcal{T}_k(s) = s$ for $|s| \leq \frac{k}{2}$ and $\mathcal{T}_k(s) = sign(s)k$ for $|s| > k$.
Taking in (27) the test function $\mathcal{T}_k^{'}(v_n)\varphi$ where $\varphi \in \mathcal{C}_c^\infty(\mathcal{Q}_T)$ we get

$$\begin{aligned}(\mathcal{T}_k(v_n))_t &- \text{div}(\mathcal{T}_k^{'}(v_n)|\nabla u_n|^{p-2}\nabla u_n) + |\nabla u_n|^{p-2}\nabla u_n.\nabla v_n \mathcal{T}_k^{''}(v_n)\\ &+g(u_n)|\nabla u_n|^p\mathcal{T}_k^{'}(v_n) - nT_n(v_n^-)\mathcal{T}_k^{'}(v_n)\end{aligned}$$

$$= \mathcal{T}_k^{'}(v_n)g_1^n - \text{div}(G_0^n\mathcal{T}_k^{'}(v_n)) + \mathcal{T}_k^{''}(v_n)G_0^n\nabla v_n, \ \text{div}(G_0^n) = g_0^n. \tag{44}$$

$\mathcal{T}_k^{'}$ has a compact support, thanks to (31) and Lemma 3.1 the quantities $|\nabla u_n|^{p-2}\nabla u_n.\nabla v_n\mathcal{T}_k^{''}(v_n)$, $nT_n(v_n^-)\mathcal{T}_k^{'}(v_n)$ and $\mathcal{T}_k^{''}(v_n)G_1^n\nabla v_n$ are bounded in $L^1(\mathcal{Q}_T)$, so are $g(u_n)|\nabla u_n|^p\mathcal{T}_k^{'}(v_n)$ and $\mathcal{T}_k^{'}(v_n)g_1^n$ because of (19) and (30). Similarly, $G_0^n\mathcal{T}_k^{'}(v_n)$ and $\mathcal{T}_k^{'}(v_n)|\nabla u_n|^{p-2}\nabla u_n$ are bounded in $(L^{p'}(\mathcal{Q}_T))^N$, then from (44) $(\mathcal{T}_k(v_n))_t$ is bounded in $\mathcal{V}_0^{'} + L^1(\mathcal{Q}_T)$. Since $\mathcal{T}_k(v_n)$ is bounded in $\mathcal{V}_0$ a compactness result from[17] leads to compactness of $(\mathcal{T}_k(v_n))_n$ in $L^1(\mathcal{Q}_T)$. Thus for a subsequence, it also a Cauchy sequence in measure.
Take $\tau > 0$, for n and m large we have

$$|\{|\mathcal{T}_k(v_n) - \mathcal{T}_k(v_m)| > \tau\}| \leq \varepsilon. \tag{45}$$

By the choice of $\mathcal{T}_k$:

$$|\{|v_n - v_m| > \tau\}| \leq |\left\{|v_n| > \frac{k}{2}\right\}| + |\left\{|v_m| > \frac{k}{2}\right\}| + |\{|\mathcal{T}_k(v_n) - \mathcal{T}_k(v_m)| > \tau\}|. \tag{46}$$

Then from (43):

$$|\{|v_n - v_m| > \tau\}| \leq 3\varepsilon, \tag{47}$$

so that $v_n = u_n - g_2^n$ is a Cauchy sequence in measure. Passing to subsequence and using (31) there exists a measurable function u such that u_n converges to u almost everywhere and

$$T_k(u_n - g_2^n) \rightharpoonup T_k(u - g_2) \text{ in } \mathcal{V}_0,$$

and

$$T_k(u_n) \rightharpoonup T_k(u) \text{ in } \mathcal{V}_0.$$

Moreover, Lemma 3.1 and Fatou's lemma give $v^- = 0$ a.e. in $\mathcal{Q}_T$ then $u \geq g_2$ a.e. in $\mathcal{Q}_T$.
Now we prove (40), it suffices to prove (38).
Take $\psi(v_n) = T_k^+(v_n - T_h(v_n))$ as test function in (27) we obtain

$$\ll \frac{\partial v_n}{\partial t}, \psi(v_n) \gg + \int_{\mathcal{Q}_T} |\nabla u_n|^{p-2} \nabla u_n \nabla v_n \chi_{\{h \leq v_n \leq h+k\}}$$
$$+ \int_{\mathcal{Q}_T} g(u_n) |\nabla u_n|^p \psi(v_n) - n \int_{\mathcal{Q}_T} T_n(v_n^-) \psi(v_n) = \int_{\mathcal{Q}_T} g_1^n \psi(v_n)$$
$$+ \int_0^T \langle g_0^n, \psi(v_n) \rangle$$

the truncation function satisfies the sign condition and ψ is positive then

$$\int_{\mathcal{Q}_T} g(u_n)|\nabla u_n|^p \psi(v_n) \geq 0 \text{ and } n \int_{\mathcal{Q}_T} T_n(v_n^-)\psi(v_n) = 0,$$

and due to the integration result in Proposition 3.2 we have

$$\begin{aligned} &\int_{\mathcal{Q}_T} |\nabla u_n|^p \chi_{\{h \leq v_n \leq h+k\}} - \int_{\mathcal{Q}_T} |\nabla u_n|^{p-2} \nabla u_n \nabla g_2^n \chi_{\{h \leq v_n \leq h+k\}} \\ &\leq \int_{\{v_n \geq h\}} |g_1^n| + \int_{\mathcal{Q}_T} |G_0^n| |\nabla \psi(v_n)| \end{aligned} \tag{48}$$

then

$$C \int_{\{h \leq v_n \leq h+k\}} |\nabla u_n|^p \leq \int_{\{h \leq v_n \leq h+k\}} |\nabla G_0^n|^{p'} + \int_{\{v_n \geq h\}} |g_1^n| + \int_{\{h \leq v_n \leq h+k\}} |\nabla g_2^n|^p$$

by (37) and the equi-integrability of the sequences $(g_1^n)_n$, $(|G_0^n|^{p'})_n$ and $(|\nabla g_2^n|^p)_n$ in $L^1(\mathcal{Q}_T)$ we obtain (38), The boundedness of $(T_{h+k}(v_n))_n$ in $\mathcal{V}_0$ and (38) gives (40). $\square$

Return to the proof of Theorem 3.1. The purpose in what follows is the strong convergence

$$T_k(v_n) \to T_k(v) \text{ in } \mathcal{V}_0 \tag{49}$$

to this end we use almost the same techniques as in[8] and.[13] Recall first the following time regularization of $T_k(v)$ due to R. Landes in the stationary case and adapted to the parabolic one in[13] and.[8]
Let $(z_\nu)_\nu$ be a sequence of positive function such that:

$$\begin{aligned}
&z_\nu \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega),\ \|z_\nu\|_\infty \le k,\ \forall \nu > 0,\\
&z_\nu \to T_k(v)(T) \text{ a.e. in } \Omega \text{ as } \nu \text{ tends to infinity},\\
&\lim_{\nu \to +\infty} \frac{1}{\nu}\|z_\nu\|_{W_0^{1,p}(\Omega)} = 0,
\end{aligned}$$

then note $T_k(v)_\nu$ the unique solution of the problem:

$$\begin{cases} \dfrac{\partial T_k(v)_\nu}{\partial t} = \nu(T_k(v) - T_k(v)_\nu), \\ T_k(v)_\nu(0) = z_\nu \end{cases}$$

which has the form

$$T_k(v)_\nu(t) = \int_0^t \nu exp(\nu(s-t))v(s)ds + z_\nu exp(-\nu t) \tag{50}$$

Then we have (see[11])

$$\begin{aligned}
&T_k(v)_\nu \to T_k(v) \text{ in } \mathcal{V}_0 \text{ and a.e. in } \mathcal{Q}_T.\\
&\|T_k(v)_\nu\|_\infty \le k,\ \forall \nu > 0.
\end{aligned}$$

We are ready to prove (49).
Let $(u_n)_n$ be a sequence of solutions of (27) where μ_n satisfies (28), and let u given by Proposition 3.3. For $h > 2k$, we introduce as in[13] and[8] the following function

$$w_n = T_{2k}(v_n - T_h(v_n) + T_k(v_n) - T_k(v)_\nu)$$

we will note $\varepsilon(n, \nu, h)$ all the quantities (possibly different) such that

$$\lim_{h \to +\infty} \lim_{\nu \to +\infty} \lim_{n \to +\infty} \varepsilon, \nu, h) = 0$$

and this will be the order in which the parameters we use will tends to infinity, that is, first n, then ν and finally h. Choosing w_n as test function in (27) we have

$$\begin{aligned}
&\ll \frac{\partial v_n}{\partial t}, w_n \gg + \int_{\mathcal{Q}_T} |\nabla u_n|^{p-2}\nabla u_n \nabla w_n + \int_{\mathcal{Q}_T} g(u_n)|\nabla u_n|^p w_n\\
&-n\int_{\mathcal{Q}_T} T_n(v_n^-) w_n = \int_{\mathcal{Q}_T} g_1^n w_n + \ll g_0^n, w_n \gg
\end{aligned}$$

remark that

$$w_n = T_{2k}(T_M(v_n) - T_h(v_n) + T_k(v_n) - T_k(v)_\nu)$$

where $M = h + 2k$, then $(w_n)_n$ is bounded in $\mathcal{V}_0$ and

$$w_n \rightharpoonup T_{2k}(v - T_h(v) + T_k(v) - T_k(v)_\nu) \text{ in } \mathcal{V}_0 \text{ and a.e. in } \mathcal{Q}_T$$

and

$$\lim_{n\to+\infty} \int_{\mathcal{Q}_T} g_1^n w_n = \int_{\mathcal{Q}_T} g_1 T_{2k}(v - T_h(v) + T_k(v) - T_k(v)_\nu),$$
$$\lim_{n\to+\infty} \ll g_0^n, w_n \gg = \ll g_0, T_{2k}(v - T_h(v) + T_k(v) - T_k(v)_\nu) \gg$$

the fact that $T_k(v)_\nu \to T_k(v)$ in $\mathcal{V}_0$ and a.e. in $\mathcal{Q}_T$ gives

$$\lim_{\nu\to+\infty} \int_{\mathcal{Q}_T} g_1 T_{2k}(v - T_h(v) + T_k(v) - T_k(v)_\nu) = \int_{\mathcal{Q}_T} g_1 T_{2k}(v - T_h(v)),$$
$$\lim_{\nu\to+\infty} \ll g_0, T_{2k}(v - T_h(v) + T_k(v) - T_k(v)_\nu) \gg = \ll g_0, T_{2k}(v - T_h(v)) \gg$$

by the Lebesgue theorem

$$\lim_{h\to+\infty} \int_{\mathcal{Q}_T} g_1 T_{2k}(v - T_h(v)) = 0.$$

Moreover

$$\ll g_0, T_{2k}(v - T_h(v)) \gg = \int_{\{h\le v\le h+2k\}} G_0 \nabla v \le \|G_0\|_{p'} \left(\int_{\{h\le v\le h+2k\}} |\nabla v|^p \right)^{\frac{1}{p}},$$

then, because of (40) we have

$$\ll \frac{\partial v_n}{\partial t}, w_n \gg + \int_{\mathcal{Q}_T} |\nabla u_n|^{p-2} \nabla u_n \nabla w_n - n \int_{\mathcal{Q}_T} T_n(v_n^-) w_n = \varepsilon(n, \nu, h)$$

but

$$\begin{aligned}
& n \int_{\mathcal{Q}_T} T_n(v_n^-) w_n \\
&= n \int_{\mathcal{Q}_T} T_n(v_n^-) T_{2k}(v_n^+ - v_n^- - T_h(v_n^+ - v_n^-) + T_k(v_n^+ - v_n^-) - T_k(v)_\nu) \\
&= n \int_{\mathcal{Q}_T} T_n(v_n^-) T_{2k}(-v_n^- - T_h(-v_n^-) + T_k(-v_n^-) - T_k(v)_\nu) \\
&= n \int_{\mathcal{Q}_T} T_n(v_n^-) T_{2k}(T_h(-v_n^-) - v_n^- - (T_k(v_n^-) + T_k(v)_\nu))
\end{aligned}$$

The positivity of v and (50) gives $T_k(v)_\nu \geq 0$ then $T_k(v_n^-) + T_k(v)_\nu \geq 0$, and clearly $T_h(-v_n^-) - v_n^- \leq 0$ a.e. in $\mathcal{Q}_T$, then $n\int_{\mathcal{Q}_T} T_n(v_n^-)w_n \leq 0$ and finally

$$\ll \frac{\partial v_n}{\partial t}, w_n \gg + \int_{\mathcal{Q}_T} |\nabla u_n|^{p-2}\nabla u_n \nabla w_n \leq \varepsilon(n, \nu, h)$$

the rest of the proof is the same as[13] as far as the term $\ll \frac{\partial v_n}{\partial t}, w_n \gg$ is concerned and in[8] for the others.
Periodicity condition passes to the limit in the following way:
By (44) one has

$$\partial_t S(u_n - g_2^n) \text{ converge strongly in } \mathcal{V}_0' + L^1(\mathcal{Q}_T),$$

but

$$S(u_n - g_2^n) \text{ converge strongly in } \mathcal{V}_0.$$

Then by Lemma 3.2

$$S(u_n - g_2^n) \to S(u - g_2) \text{ strongly in } C([0,T]; L^1(\Omega)).$$

But $S(u_n - g_2^n)(0) = S(u_n - g_2^n)(T)$ then

$$S(u - g_2)(0) = S(u - g_2)(T) \text{ in } \Omega.$$

And periodicity is now proved. □

References

1. N. Alaa, M. Iguernane, *Weak periodic solutions of some quasilinear parabolic equations with data measures* , Journal of Inequalities in Pure and Applied Mathematics. **3** (3),No. 46, (2002).
2. N. Alaa, M. Pierre, *Weak solutions of some quasilinear elliptic equations with data measures*, SIAM J. Math.
   Anal. **24** (1993), n. 1, 23-35.
3. P. Benilan, L. Boccardo, T. Gallouët, R. Gariepy, M. Pierre, J. L. Vàzquez, *An L^1 theory of existence and uniqueness of nonlinear elliptic equations*, Ann. Scuola Norm. Sup. pisa CI. Sci., **22**, (1995),641-655.
4. D. Blanchard, A. Porretta, *Nonlinear parabolic equations with natural growth terms and measure initial data.* Annali della Scuola Normale Superiore di Pisa - Classe di Scienze Sér. 4, **30** . 3-4 (2001), p. 583-622
5. D. Blanchard, F. Murat, *Renormalised solutions of nonlinear parabolic problems with L1-data: existence and uniqueness*, Proc. Royal Soc. Edinburgh Sect. A **127** (1997), 1137-1152.

6. L. Boccardo, S. Segura, C. Trombetti, Bounded and unbounded solutions for a class of quasi-linear elliptic problems with a quadratic gradient term, *J. Math. Pures Appl.*, (9) **80** (2001), 919–940.
7. G. Dal Maso, F. Murat, L. Orsina, A. Prignet,*Definition and existence of renormalized solutions of elliptic equations with general measure data*, C. R. Acad. Sci. Paris Sér. I Math. 325 (1997), 481-486.
8. J. Droniou, A. Porretta, A. Prignet, *Parabolic capacity and soft measures for nonlinear equations*, Potential Analysis, Volume **19**, Number 2, September 2003, pp. 99-161(63)
9. A. El Hachimi And A. Lamrani Alaoui, *Existence of stable periodic solutions for quasilinear parabolic problems in the presence of well-ordered lower and upper-solutions*,proceeding of EJDE, **9** (2002), pp. 1-10.
10. P. Hess. *periodic-parabolic boundary value problem and positivity*, Pitman Res. Notes Mathematics, Vol. 247, Longman Sci. Tech., Harlow, 1991.
11. R. Landes, *On the existence of weak solutions for quasilinear parabolic boundary value problems*, Proc. Roy. Soc. Edinburgh Sect. A 89 (1981), 217-237.
12. P. Polacik, Parabolic equations: asymptotic behaviour and dynamics on invariant manifolds, In: Handbook of Dyn. Syst., Vol. 2, North-Holland, Amsterdam, pp. 835–883.
13. A. Porretta, *Existence results for nonlinear parabolic equations via strong convergence of truncations* , Ann. Mat. Pura ed Applicata (IV) **177** (1999), pp. 143–172.
14. A. Porretta; *Nonlinear equations with natural growth terms and measures data*, proceeding of EJDE, **9** (2002), pp. 183-202.
15. J. M. Rakotoson, A compactness lemma for quasilinear problems: Application to parabolic equations, J. Funct. Anal., Vol. 106, No. **2**, pp. 1163-1175, 1992.
16. S. Segura de Leon, *Existence and uniqueness for L^1 data of some elliptic equations with natural growth*, Advances in Diff. Eq., to appear.
17. J. Simon, *compact sets in $L^p(0,T;B)$*, Ann. Math. Pura Appl. , **146** (4) (1987), 65-96.

Existence and L^∞-regularity results for some nonlinear elliptic Dirichlet problems

Ahmed Youssfi

Department of Mathematics and Informatic
Faculty of Sciences Dhar El Mahraz
B.P 1796 Atlas, Fès, Morocco
E-mail: ahmed.youssfi@caramail.com

We summarize recent lectures devoted to the study of the existence and L^∞-regularity results of some Dirichlet problems associated to equations having degenerated coercivity in the principal part.

Keywords: Orlicz spaces; Nonlinear elliptic equations; Degenerate coercivity; A priori estimates; L^∞-estimates; Rearrangements.

1. Introduction

During the 19th century, Hölder regularity results to the Dirichlet problem for linear elliptic equations has been object of much research and has developed by many authors. Far from being complete, the story goes that motivated in solving the Hilbert's 19th conjecture, E. DeGiorgi[11] in 1957, the first, proved his famous regularity result which asserts that local weak solutions for the linear problem

$$\begin{cases} (a_{i,j}u_{x_i})_{x_j} = 0 \quad \text{weakly in } \Omega, \\ u \in W_0^{1,2}(\Omega), \end{cases} \tag{1}$$

where the coefficients $x \to a_{i,j}(x), i, j = 1, ..., N$ are only bounded and measurable and satisfying the uniform ellipticity condition

$$a_{i,j}\xi_i\xi_j \geq \alpha|\xi|^2, \ \text{ a.e. in } \Omega, \ \ \forall \xi \in I\!R^N, \text{ for some } \alpha > 0, \tag{2}$$

are Hölder continuous by studying local pointwise estimates. The global bound appears in the 1960's in the works of G. Stampacchia[15,16] and Ladyzhenskaja-Ural'tseva.[13] Following the methods of DeGiorgi, Ladyzhenskaja and Ural'tzeva established that weak solutions of quasilinear elliptic equations are Hölder continuous. Since, several $L^\infty-$regularity results to the Dirichlet problem, linear or nonlinear, more general than (1) have been established.

Let us consider the problem

$$\begin{cases} -\text{div}(a(x,u)\nabla u) = f \quad \text{in } \Omega, \\ u \in H_0^1(\Omega) \cap L^\infty(\Omega), \end{cases} \tag{3}$$

with the degenerate coercivity

$$a(x,s) \geq \frac{\alpha}{(1+s)^\theta} \text{ with } \alpha > 0$$

where Ω is a bounded open subset of $\mathbb{R}^N$, $N \geq 2$, and

$$f \in L^m(\Omega) \text{ with } m > N/2.$$

The existence of bounded solutions for (3) was proved in Alvino et al.,[2] Boccardo-Brezis[6] and in Boccardo et al.[7] The result was then extended in Alvino et al.[1] for the nonlinear elliptic problem

$$\begin{cases} -\text{div} a(x,u,\nabla u) = f \quad \text{in } \Omega, \\ u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega), \end{cases} \tag{4}$$

$p > 1$ with degenerate coercivity

$$a(x,s,\xi) \cdot \xi \geq \frac{\alpha}{(1+s)^{\theta(p-1)}} |\xi|^p \text{ with } \alpha > 0 \tag{5}$$

and where

$$f \in L^m(\Omega) \text{ with } m > N/p. \tag{6}$$

When we replace in (4) f by $-\text{div} g$, where

$$|g| \in L^m(\Omega) \text{ with } m > N/(p-1), \tag{7}$$

Benkirane-Youssfi[5] have proved that problem (4) has bounded solutions. Strongly nonlinear elliptic problems are also treated. It is so the problem

$$\begin{cases} A(u) + H(x,u,\nabla u) = f - \text{div} g \quad \text{in } \Omega, \\ u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega), \end{cases} \tag{8}$$

where $A(u) = -\text{div} a(x,u,\nabla u)$ has uniform coercivity and f and g satisfying (6) and (7) respectively, has at least one solution Boccardo et al.[8] and Ferone et al.[10] In Boccardo et al.[9] the authors have considered the problem (8) with $A(u) = -\text{div}(a(x,u)\nabla u)$ having degenerate coercivity and $g = 0$ and have shown that it has at least one solution in $H_0^1(\Omega) \cap L^\infty(\Omega)$. The general case $1 < p < +\infty$ with $g = 0$ and $a(x,s,\xi)$ satisfying (5) Trombetti[21] established that (8) has at least one solution.

Here we present a survey on some recent existence and L^∞-regularity results for the Dirichlet problem associated to nonlinear elliptic equations on bounded open subset Ω of $\mathbb{R}^N, N \geq 2$, involving the following non-everywhere nonlinear differential operator of Leray-Lions type

$$A(u) := -\operatorname{div} a(x, u, \nabla u).$$

The function a may have nonstandard growth and satisfies the condition:

$$a(x, s, \xi)\cdot\xi \geq \overline{M}^{-1}(M(h(|s|)))M(|\xi|), \tag{9}$$

where M is an N-function (take as example $M(t) = t^p, p > 1$) and $h : \mathbb{R}^+ \to]0, +\infty[$ is a continuous decreasing function such that : $0 < h(0) \leq 1$ and its primitive

$$H(s) = \int_0^s h(t)dt \tag{10}$$

is unbounded (take for instance $h(t) = \frac{1}{(1+t)^\theta}, 0 \leq \theta \leq 1$.)

It is worth recalling that in the L^p setting, a priori bounds for solutions of nonlinear elliptic Dirichlet problems can be derived by using either the method of Stampacchia[15] or by means of the rearrangements techniques widely developed in Talenti.[17–20]

Observe that because of the lack of coercivity generated by the assumption (9), the operator A degenerates when its second argument has large values. Therefore, the classical existence results in Leray-Lions[14] and Gossez-Mustonen[12] used to prove the existence of a solution for the problem (4) cannot be applied even if the datum f is very regular. To get rid of this difficulty, we will consider approximate equations in which we introduce a truncation, in order to get non degenerate approximated problems which thus have solution. Once this done, some a priori estimates on the solutions of these problems are proved. To complete the ingredients which allow us to pass to the limit, we prove the almost everywhere convergence of the gradients of solutions.

2. Standard growth

One stands in the L^p setting. Let $M(t) = t^p$ with $1 < p < N$. It is known that the Dirichlet problem

$$\begin{cases} A(u) = f \text{ in } \Omega, \\ u = 0 \quad\quad \text{on } \partial\Omega, \end{cases} \tag{11}$$

has bounded solutions provided that the datum f satisfies (6). Even in the case when h is the constant function, the limit case $f \in L^{\frac{N}{p}}$ yields a solution

u in a suitable Orlicz space of exponential type and it is (in general) not bounded. Precisely, $H(u) \in L_\phi(\Omega)$ with $\phi(t) = \exp(t) - t - 1$ and H is the function defined in (10).
We consider a subclass of $L^{\frac{N}{p}}(\Omega)$ larger than (6) which guaranties the boundedness of solutions of the problem (11).
In addition of (9), assume that the Carathéodory function $a : \Omega \times I\!R \times I\!R^N \to I\!R^N$ satisfies for almost x in Ω, for every $s \in I\!R$ and for every $\xi, \eta \in I\!R^N$ with $\xi \neq \eta$

$$|a(x,s,\xi)| \leq a_0(x) + |s|^{p-1} + |\xi|^{p-1} \tag{12}$$

where a_0 is a non negative function in $L^{p'}(\Omega)$ with $p' = \frac{p}{p-1}$, and

$$(a(x,s,\xi) - a(x,s,\eta)).(\xi - \eta) > 0. \tag{13}$$

Let us denote by $L^{\frac{N}{p}} \log^\alpha L$ the Orlicz space generated by the N-function $\Theta(t) = t^{\frac{N}{p}} \log^\alpha(e+t)$. Assume that

$$f \in L^{\frac{N}{p}} \log^\alpha L \quad \text{with} \quad \alpha > \frac{N(p-1)}{p}. \tag{14}$$

Our first result is the following

Theorem 2.1 (Benkirane-Youssfi-Meskine,[3] 2008).
Suppose that (9), (12), (13) and (14) are filled. Then, the problem (11) has at least one solution u in $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ in the sense that

$$\int_\Omega a(x,u,\nabla u) \cdot \nabla v dx = \int_\Omega f v dx \tag{15}$$

for all v in $\mathcal{D}(\Omega)$.

Proof. We only give a sketch. Denote by T_k, the truncation at level $k > 0$, defined as $T_k(s) = \max(-k, \min(s,k))$ and set $G_k(s) = s - T_k(s)$. To get rid of the lack of coerciveness of the operator A, we introduce the truncation operator. Let $\{f_n\}$ be a sequence of L^∞-functions such that

$$f_n \to f \quad \text{in} \quad L^1(\Omega) \quad \text{and} \quad |f_n| \leq |f|. \tag{16}$$

The approximate problem

$$\begin{cases} u_n \in W_0^{1,p}(\Omega) \quad \text{such that} \\ \displaystyle\int_\Omega a(x, T_n(u_n), \nabla u_n) \cdot \nabla \phi \, dx = \int_\Omega f_n \phi \, dx \\ \quad \text{holds for every } \phi \quad \text{in } W_0^{1,p}(\Omega) \end{cases} \tag{17}$$

admits at least one solution thanks to the Leray-Lions existence theorem (see Leray-Lions[14]). For $t > 0$ and $\epsilon > 0$, we use $T_\epsilon(G_t(u_n))$ as test function in (17). Being h decreasing, letting ϵ tends to 0^+ we arrive at

$$h^{p-1}(t)\left(-\frac{d}{dt}\int_{\{|u_n|>t\}}|\nabla u_n|^p dx\right) \le \int_{\{|u_n|>t\}}|f|dx. \tag{18}$$

On the other hand, Hölder's inequality enables us to get

$$-\frac{d}{dt}\int_{\{|u_n|>t\}}|\nabla u_n|dx \le (-\mu'(t))^{\frac{1}{p'}}\left(-\frac{d}{dt}\int_{\{|u_n|>t\}}|\nabla u_n|^p dx\right)^{\frac{1}{p}}, \tag{19}$$

where μ_n denotes the distribution function of u_n, that is

$$\mu_n(t) = |\{x \in \Omega : |u_n(x)| > t\}|.$$

Combining (18), (19) and the following well known inequality (see Talenti[20])

$$NC_N^{\frac{1}{N}}(\mu_n(t))^{1-\frac{1}{N}} \le -\frac{d}{dt}\int_{\{|u_n|>t\}}|\nabla u_n|dx,$$

we obtain

$$h(t) \le \frac{-\mu_n'(t)}{N^{p'}C_N^{\frac{p'}{N}}(\mu_n(t))^{p'(1-\frac{1}{N})}}\left(\int_{\{|u_n|>t\}}|f|dx\right)^{\frac{p'}{p}}.$$

In the above inequality, we use Hölder's inequality in Orlicz spaces, then we integrate both sides between 0 and τ obtaining by the definition of the rearrangement

$$H(u_n^*(\sigma)) \le \frac{2^{\frac{p'}{p}}\|f\|^{\frac{p'}{p}}_{L^{\frac{N}{p}}\log^\alpha L}}{N^{p'}C_N^{\frac{p'}{N}}}\int_{\frac{1}{|\Omega|}}^{\frac{1}{\sigma}}\frac{ds}{s\log^{\alpha\frac{p'}{N}}(e+s)}.$$

Thus, by (14), the sequence $\{u_n\}$ is uniformly bounded in $L^\infty(\Omega)$. It is now easy to get an estimation of $\{u_n\}$ in the energy space $W_0^{1,p}(\Omega)$ which implies that there is a function u such that

$$u_n \to u \quad \text{weakly in } W_0^{1,p}(\Omega) \text{ and a.e in } \Omega. \tag{20}$$

Then, we prove that

$$\nabla u_n \to \nabla u \qquad \text{a.e in } \Omega.$$

Which enables us to pass to the limit in (17) to get (15). □

3. Nonstandard growth

Here, we do not assume the Δ_2-condition on the N-function M. In particular, the corresponding Orlicz-Sobolev spaces are not reflexive. We extend the class of functions satisfying (6) to the Orlicz-Sobolev setting by assuming one of the following two assumptions: Either

$$f \in L^N(\Omega), \tag{21}$$

or

$$f \in L^m(\Omega) \ \text{ and } \int_{\cdot}^{+\infty} \left(\frac{t}{M(t)}\right)^{\frac{m}{N-m}} dt < +\infty. \tag{22}$$

In addition of (9) and (13), we make the following assumption:

$$|a(x,s,\xi)| \leq a_0(x) + k_1 \overline{P}^{-1} M(k_2|s|) + k_3 \overline{M}^{-1} M(k_4|\xi|) \tag{23}$$

where $a_0(x)$ belongs to $E_{\overline{M}}(\Omega)$, P is an N-function such that $P \ll M$ and k_1, k_2, k_3, k_4 are constants in $\mathbb{R}_+^*$. We prove the following

Theorem 3.1 (Youssfi,[22] 2007). *Suppose that (9), (13) and (23) are filled. Under either (21) or (22), the problem (11) has at least one solution u in $W_0^1 L_M(\Omega) \cap L^\infty(\Omega)$ in the sense that*

$$\int_\Omega a(x,u,\nabla u) \cdot \nabla v dx = \int_\Omega f v dx \tag{24}$$

for all v in $\mathcal{D}(\Omega)$.

Remark 3.1. The previous theorem is surprising, since the function h does not influence the result. This seems to be natural, since if one looks for bounded solutions the degeneracy of the operator A "disappears". Our result includes classical similar results, notably the G. Stampacchia's one.

We extend the result of the previous theorem 3.1 to the strongly nonlinear problem

$$\begin{cases} A(u) + H(x,u,\nabla u) = f \text{ in } \Omega, \\ u = 0 \qquad\qquad\qquad\qquad \text{on } \partial\Omega, \end{cases} \tag{25}$$

where H is a Carathéodory function which doesn't satisfy necessarily the famous sign condition $H(x,s,\xi)s \geq 0$, but only the following growth condition

$$|H(x,s,\xi)| \leq \beta(s) M(|\xi|), \tag{26}$$

where $\beta : I\!R \rightarrow I\!R^{+}$ is a continuous function such that the function

$$t \rightarrow \frac{\beta(t)}{\overline{M}^{-1}(M(h(|t|)))}$$

belongs to $L^{1}(I\!R)$. So by defining

$$\gamma(s) = \int_{0}^{s} \frac{\beta(t)}{\overline{M}^{-1}(M(h(|t|)))} dt$$

for all $s \in I\!R$, the function γ is bounded.

Theorem 3.2 (Benkirane-Youssfi,[4] 2008). *Suppose that (9), (13), (23) and (26) are filled. Under either (21) or (22), the problem (25) has at least one solution u in $W_{0}^{1}L_{M}(\Omega) \cap L^{\infty}(\Omega)$ in the sense that*

$$\int_{\Omega} a(x, u, \nabla u) \cdot \nabla v dx + \int_{\Omega} H(x, u, \nabla u) v dx = \int_{\Omega} f v dx \tag{27}$$

for all $v \in W_{0}^{1}L_{M}(\Omega) \cap L^{\infty}(\Omega)$.

Remark 3.2. In addition of the degeneration of the operator A, the fact that H does not satisfy the sign condition creates another problem of getting the a priori estimates. To overcome this hindrance, we use test functions of exponential type containing the function γ. Note that the result of Theorem 3.2 extends those obtained in Boccardo et al.[9] and Trombetti.[21]

Proof. Summarized.
Step 1: A priori estimates: We define A_n and B_n as

$$A_n(u) := -\operatorname{div} a(x, T_n(u), \nabla u)$$

and

$$B_n(x, u, \nabla u) = T_n(B(x, u, \nabla u)).$$

Denote by m^{*} either N or m according as we assume (21) or (22), and let $\{f_n\} \subset W^{-1}E_{\overline{M}}(\Omega)$ be a sequence of smooth functions such that

$$f_n \rightarrow f \ \text{ strongly in } \ L^{m^{*}}(\Omega)$$

and

$$\|f_n\|_{m^{*}} \leq \|f\|_{m^{*}}.$$

By Proposition 2 of Gossez-Mustonen,[12] there exists at least one solution $u_n \in D(A_n + B_n) \subset W_{0}^{1}L_{M}(\Omega)$ to the approximate equation

$$-\operatorname{div} a(x, T_n(u_n), \nabla u_n) \, + \, B_n(x, u_n, \nabla u_n) \, = \, f_n$$

in the sense that

$$\int_\Omega a(x,T_n(u_n),\nabla u_n)\cdot\nabla v dx + \int_\Omega B_n(x,u_n,\nabla_n)v dx = \int_\Omega f_n v dx \quad (28)$$

for all $v \in W_0^1L_M(\Omega)$.

Lemma 3.1. *Let u_n be a solution of (28). For all t,ϵ in $\mathbb{R}_+^*$, one has the following inequalities:*

$$\begin{aligned}&\int_{\{t<u_n\le t+\epsilon\}} a(x,T_n(u_n),\nabla u_n)\cdot\nabla u_n e^{\gamma(u_n^+)}dx\\ &\le \int_{\{u_n>t\}} f_n^+ e^{\gamma(u_n^+)}T_\epsilon(G_t(u_n^+))dx.\end{aligned} \quad (29)$$

$$\begin{aligned}&\int_{\{-t-\epsilon<u_n\le -t\}} a(x,T_n(u_n),\nabla u_n)\cdot\nabla u_n e^{\gamma(u_n^-)}dx\\ &\le \int_{\{u_n<-t\}} f_n^- e^{\gamma(u_n^-)}T_\epsilon(G_t(u_n^-))dx.\end{aligned} \quad (30)$$

Proof. To prove (29) we use the test function

$$e^{\gamma(T_k(u_n^+))}T_\epsilon(G_t(T_k(u_n^+)))$$

that belongs to $W_0^1L_M(\Omega)$ for all $k>0$ and we use the function

$$-e^{\gamma(T_k(u_n^-))}T_\epsilon(G_t(T_k(u_n^-)))$$

which belongs to $W_0^1L_M(\Omega)$ to prove (30). □

Lemma 3.2. *There exists a constant c_0, not depending on n, such that for almost every $t>0$*

$$-\frac{d}{dt}\int_{\{|u_n|>t\}} \overline{M}^{-1}(M(h(|u_n|)))M(|\nabla u_n|)dx \le c_0\int_{\{|u_n|>t\}}|f_n|dx. \quad (31)$$

Proof. The two functions $e^{\gamma(u_n^+)}$ and $e^{\gamma(u_n^-)}$ are bounded in $L^\infty(\Omega)$, we sum up both inequalities (29) and (30) obtaining (31). □

The following comparison lemma constitutes the crucial step in the proof.

Lemma 3.3. *Let $K(t)=\frac{M(t)}{t}$ and $\mu_n(t)=|\{x\in\Omega : |u_n(x)|>t\}|$, for all $t>0$. We have for almost every $t>0$:*

$$h(t) \le \frac{2M(1)(-\mu_n'(t))}{\overline{M}^{-1}(M(1))NC_N^{\frac{1}{N}}\mu_n(t)^{1-\frac{1}{N}}}K^{-1}\left(\frac{c_0\int_{\{|u_n|>t\}}|f_n|dx}{\overline{M}^{-1}(M(1))NC_N^{\frac{1}{N}}\mu_n(t)^{1-\frac{1}{N}}}\right) \quad (32)$$

where C_N stands for the measure of the unit ball in $I\!R^N$ and c_0 is the constant which appears in (31).

Proof. The function $C(t) = \dfrac{1}{K^{-1}(t)}$ is decreasing and convex (see Talenti[19]). Hence, Jensen's inequality yields

$$C\left(\frac{\displaystyle\int_{\{t<|u_n|\le t+k\}} \overline{M}^{-1}(M(h(|u_n|)))M(|\nabla u_n|)dx}{\displaystyle\int_{\{t<|u_n|\le t+k\}} \overline{M}^{-1}(M(h(|u_n|)))|\nabla u_n|dx}\right)$$

$$= C\left(\frac{\displaystyle\int_{\{t<|u_n|\le t+k\}} K(|\nabla u_n|)\overline{M}^{-1}(M(h(|u_n|)))|\nabla u_n|dx}{\displaystyle\int_{\{t<|u_n|\le t+k\}} \overline{M}^{-1}(M(h(|u_n|)))|\nabla u_n|dx}\right)$$

$$\le \frac{\displaystyle\int_{\{t<|u_n|\le t+k\}} \overline{M}^{-1}(M(h(|u_n|)))dx}{\displaystyle\int_{\{t<|u_n|\le t+k\}} \overline{M}^{-1}(M(h(|u_n|)))|\nabla u_n|dx}$$

$$\le \frac{\overline{M}^{-1}(M(h(t)))(-\mu_n(t+k)+\mu_n(t))}{\overline{M}^{-1}(M(h(t+k)))\displaystyle\int_{\{t<|u_n|\le t+k\}} |\nabla u_n|dx}.$$

Taking into account that $\overline{M}^{-1}(M(h(t))) \le \overline{M}^{-1}(M(1))$, using the convexity of C and then letting $k \to 0^+$, we obtain for almost every $t > 0$

$$\frac{\overline{M}^{-1}(M(1))}{\overline{M}^{-1}(M(h(t)))}\, C\left(\frac{-\dfrac{d}{dt}\displaystyle\int_{\{|u_n|>t\}} \overline{M}^{-1}(M(h(|u_n|)))M(|\nabla u_n|)dx}{\overline{M}^{-1}(M(1))(-\dfrac{d}{dt}\displaystyle\int_{\{|u_n|>t\}} |\nabla u_n|dx)}\right)$$

$$\le \frac{-\mu_n'(t)}{-\dfrac{d}{dt}\displaystyle\int_{\{|u_n|>t\}} |\nabla u_n|dx}.$$

Recall the following inequality, (see Talenti[19]):

$$-\frac{d}{dt}\int_{\{|u_n|>t\}} |\nabla u_n|dx \ge NC_N^{\frac{1}{N}}\mu_n(t)^{1-\frac{1}{N}} \quad \text{for almost every } t>0. \qquad (33)$$

The monotonicity of the function C, (31) and (33) yield

$$\frac{1}{\overline{M}^{-1}(M(h(t)))}$$
$$\leq \frac{-\mu_n'(t)}{\overline{M}^{-1}(M(1))NC_N^{\frac{1}{N}}\mu_n(t)^{1-\frac{1}{N}}} K^{-1}\left(\frac{c_0\int_{\{|u_n|>t\}}|f_n|dx}{\overline{M}^{-1}(M(1))NC_N^{\frac{1}{N}}\mu_n(t)^{1-\frac{1}{N}}}\right).$$

Using the inequality

$$M(t) \leq t\overline{M}^{-1}(M(t)) \quad \text{for all } t \geq 0 \tag{34}$$

and the fact that $0 < h(t) \leq 1$, we obtain (32). □

Step 2: L^∞-bound: If we are under (21) we get

$$\|u_n\|_\infty \leq H^{-1}\left(\frac{2M(1)}{\overline{M}^{-1}(M(1))NC_N^{\frac{1}{N}}}K^{-1}\left(\frac{c_0\|f\|_N}{\overline{M}^{-1}(M(1))NC_N^{\frac{1}{N}}}\right)N|\Omega|^{\frac{1}{N}}\right) \tag{35}$$

and if we are under (22) we get

$$\|u_n\|_\infty \leq$$

$$H^{-1}\left(\frac{2M(1)c_0^r\|f\|_m^r}{(\overline{M}^{-1}(M(1)))^{r+1}N^rC_N^{\frac{r+1}{N}}}\left(\frac{K^{-1}(\lambda)}{\lambda^r}+\int_{K^{-1}(\lambda)}^{+\infty}\left(\frac{s}{M(s)}\right)^r ds\right)\right). \tag{36}$$

This, implies that

$$\|u_n\|_\infty \leq c. \tag{37}$$

Step 3: Estimation in $W_0^1L_M(\Omega)$. An estimation in $W_0^1L_M(\Omega)$, is obtained by proving the following

Lemma 3.4.

$$(1) -\int_{\{0\leq u_n\}} a(x,T_n(u_n),\nabla u_n)\cdot\nabla u_n e^{\gamma(u_n^+)}dx \;\leq\; \int_\Omega f_n^+e^{\gamma(u_n^+)}u_n^+dx.$$

$$(2) -\int_{\{u_n\leq 0\}} a(x,T_n(u_n),\nabla u_n)\cdot\nabla u_n e^{\gamma(u_n^-)}dx \;\leq\; \int_\Omega f_n^-e^{\gamma(u_n^-)}u_n^-dx.$$

Proof. We use

$$e^{\gamma(u_n^+)}u_n^+ \in W_0^1L_M(\Omega)\cap L^\infty(\Omega)$$

to prove the first inequality and use

$$-e^{\gamma(u_n^-)}u_n^- \in W_0^1L_M(\Omega)\cap L^\infty(\Omega)$$

for the second. □

Summing up both inequalities (1) and (2) we get

$$\int_\Omega M(|\nabla u_n|)dx \ \le \ \frac{cc_1\|f\|_{m^*}|\Omega|^{1-\frac{1}{m^*}}}{\overline{M}^{-1}M(h(c))}. \tag{38}$$

Hence, there is a function $u \in W_0^1L_M(\Omega)$ such that

$$u_n \rightharpoonup u \quad \text{weakly in} \quad W_0^1L_M(\Omega) \quad \text{for} \quad \sigma(\Pi L_M, \Pi E_{\overline{M}}), \tag{39}$$

and

$$u_n \to u \quad \text{in} \quad E_M(\Omega) \quad \text{strongly} \quad \text{and} \quad \text{a.e.} \quad \text{in} \quad \Omega. \tag{40}$$

Step 4: Almost everywhere convergence of the gradients. We prove the following

Lemma 3.5. *The sequence* $\{a(x, T_n(u_n), \nabla u_n)\}$ *is uniformly bounded in* $(L_{\overline{M}}(\Omega))^N$.

Then, we prove that

$$\nabla u_n \to \nabla u \quad \text{a.e. in } \Omega. \tag{41}$$

and

$$a(x, T_n(u_n), \nabla u_n) \rightharpoonup a(x, u, \nabla u) \text{ weakly in } (L_{\overline{M}}(\Omega))^N \text{ for } \sigma(\Pi L_{\overline{M}}, \Pi E_M). \tag{42}$$

Step 5: Modular convergence of the gradients. We prove that

$$u_n \to u \quad \text{in} \quad W_0^1L_M(\Omega)$$

for the modular convergence.
Step 6: Equi-integrability of the non-linearities. We prove that

$$B_n(x, u_n, \nabla u_n) \to B(x, u, \nabla u) \quad \text{strongly in } L^1(\Omega). \tag{43}$$

Step 7: Passage to the limit. Testing by $v \in W_0^1L_M(\Omega)\cap L^\infty(\Omega)$, we have all ingredients to pass to the limit in (28) and obtain that u is a solution of (25) in the sense of (27).

References

1. A. Alvino, L. Boccardo, V. Ferone, L. Orsina, G. Trombetti, *Existence results for nonlinear elliptic equations with degenerate coercivity*, Ann. Mat. Pura Appl., **182**, (2003), 53–79.
2. A. Alvino, V. Ferone, G. Trombetti, *A priori estimates for a class of non uniformly elliptic equations*, Atti Semin. Mat. Fis. Univ. Modena **46**-suppl., (1998), 381–391.
3. A. Benkirane, A. Youssfi, D. Meskine, Bounded solutions for nonlinear elliptic equations with degenerate coercivity and data in an $L \log L$, Bull. Belg. Math. Soc. Simon Stevin **15** (2008), 369–375.
4. A. Benkirane, A. Youssfi, Existence of bounded solutions for a class of strongly nonlinear elliptic equations in Orlicz-Sobolev spaces, AJMAA, Vol. **5**, No. 1, Art. 7, (2008), 1–26.
5. Benkirane, A., Youssfi, A.: Regularity for solutions of nonlinear elliptic equations with degenerate coercivity, Ricerche Mat. **56**, no.2, (2007), 241–275.
6. L. Boccardo, H. Brezis, *Some remarks on a class of elliptic equations with degenerate coercivity,* Boll. Unione Mat. Ital., **6**, (2003), 521–530.
7. L. Boccardo, A. Dall'Aglio, L. Orsina, *Existence and regularity results for some nonlinear equations with degenerate coercivity*, Atti Sem. Mat. Fis. Univ. Modena, **46**-suppl., (1998), 51–81.
8. L. Boccardo, F. Murat, P.-L. Puel, L^∞ *estimate for some nonlinear elliptic partial differential equations and application to an existence result,* SIAM J. Math. Anal., (2) **23**, (1992), 326–333.
9. L. Boccardo, S. Segura de León, C. Trombetti, *Bounded and unbounded solutions for a class of quasi-linear elliptic problems with a quadratic gradient term*, J. Math. Pures Appl., **80**, (2001), 919–940.
10. V. Ferone, M. R. Posteraro, J. M. Rakotoson, L^∞*-estimates for nonlinear elliptic problems with p-growth in the gradient,* J. of Inequal. Appl., **3**, (1999), 109–125.
11. E. DeGiorgi, *Sulla differenziabilità e l'analiticità delle estremali degli integrali multipli regolari,* Mem. Accad. Sci. Torino, Cl. Sci. Fis. Mat. Nat. **3** (1957), 25–43.
12. J.-P. Gossez and V. Mustonen, *Variational inequalities in Orlicz-Sobolev spaces,* Nonlinear Anal., **11**, (1987), 379–49.
13. O. A. Ladyzhenskaya, N. Ural'tseva, *Linear and Quasilinear Elliptic Equations,* Academic Press, 1968.
14. J. Leray, J. L. Lions, *Quelques résultats de Višik sur les problèmes elliptiques non linéaires par les méthodes de Minty-Browder*, Bull. Soc. Mat. France **93**, (1965), 97–107.
15. G. Stampacchia, *Le problème de Dirichlet pour les équations elliptiques du second ordre à coefficients discontinus,* Ann. Ist. Fourier (Grenoble), **15**, (1965), 189–258.
16. G. Stampacchia, *équations elliptiques du second ordre à coefficients discontinus,* Montréal, Presses Univ. Montréal, 1966, (Séminaire de Math. sup., 16).

17. G. Talenti, *Inequalities in rearrangement invariant function spaces,* Nonlinear analysis, function spaces and applications,Vol. 5. Proceedings of the spring School held in Prague, May 23-28, 1994, 177-230. Mathematical Institute, Czech Academy of Sciences, and Prometheus Publishing House, Praha 1995.
18. G. Talenti, *Linear elliptic P.D.E's: Level sets, rearrangements and a priori estimates of solutions*, Boll. Un. Mat. Ital. **4**-B(6), (1985), 917–949.
19. G. Talenti, *Nonlinear elliptic equations, Rearrangements of functions and Orlicz spaces*, Ann. Mat. Pura Appl., **120**, 1979, IV. Ser., 159–184.
20. G. Talenti, *Elliptic equations and rearrangements,* Ann. Scuola. Norm. Sup. Pisa (4) **3**, (1976), 697–718.
21. C. Trombetti, *Non-uniformly elliptic equations with natural growth in the gradient,* Potential Analysis, **18**, (2003), 391–404.
22. A. Youssfi, Existence of bounded solutions for nonlinear degenerate elliptic equations in Orlicz spaces, *Electron. J. Diff. Eqns.*, **2007**, No. 54, (2007), 1–13.

Analysis of a new mixed formulation of the obstacle problem

Belkassem Seddoug, Ahmed Addou, and Abdeluaab Lidouh

Département de Mathématiques et Informatique, Faculté des sciences
Université Mohammed Premier, Oujda, Morocco

For the obstacle problem:

$$\begin{cases} \Delta u = f\chi_{[u>\psi]} + \Delta\psi, \\ u \geq \psi, \end{cases}$$

for which the unknowns are u and the free boundary $\partial[u > \psi]$. It is known[1] that the coincidence set $[u = \psi]$ is contained in the set $[f - \Delta\psi \geq 0]$. In this work we introduce $\mu = (f - \Delta\psi)^+ \chi_{[u>\psi]}$, that characterizes the domain of non contact,[2] and we analyze a mixed formulation of the obstacle problem where μ appears as a Lagrange multiplier. We show, in particular, that the solution μ_h of the approached (by linear finite element) mixed problem converges to μ with an order of convergence that is an $O(h)$, which is optimal seen the equivalent result on the approximation of the free boundary[3] and.[4]

Keywords: Obstacle problem; Variational inequalities; Free boundary; Mixed formulation.

1. Introduction

In this work we are interested in the following "model" obstacle problem:

$$\begin{cases} \text{Find } u \in K = \left\{v \in H_0^1(\Omega) : v \geq \psi \text{ a.e. in } \Omega\right\} \text{ such that} \\ \int_\Omega \nabla u \,(\nabla v - \nabla u)\, dx + \int_\Omega f\,(v-u)\,dx \geq 0, \text{ for every } v \in K, \end{cases} \tag{1}$$

where ψ, f and Ω are regular data.

As in,[1] problem (1) can be written:

$$\begin{cases} \Delta u = f - \mu_s \text{ in } \Omega, \\ u_{|\partial\Omega} = 0, \end{cases} \tag{2}$$

where μ_s is a positive measure whose support is contained in the coincidence set

$$I(u) = \{x \in \Omega : u(x) = \psi(x)\}\,.$$

The a priori determination of μ_s would permit to solve (1) as a Dirichlet problem.

In the same sense, it has been established in,[2] that problem (1) is equivalent to the following one:

$$\begin{cases} \text{Find } u \in H_0^1(\Omega) \text{ and } \mu \in L^2(\Omega) \text{ such that} \\ \Delta u = \mu + F^- + \Delta\psi \text{ in } H_0^1(\Omega), \\ \mu \in \partial\varphi(u), \end{cases} \tag{3}$$

where $F = f - \Delta\psi$ supposed in $L^2(\Omega)$, and $\varphi : v \longmapsto \int_\Omega F^+ v^+ dx$. With $h^+ = \max(h, 0)$, $h^- = \min(h, 0)$ for every h in $L^2(\Omega)$, so that $h = h^+ + h^-$. $\partial\varphi(u)$ designates the subdifferential of φ at u.

A fundamental aspect, from that point of view, is the characterization of the non contact domain $\Omega_+(u) = \Omega \backslash I(u)$, and therefore the free boundary, which is also an unknown of the problem, with the help of μ.

What we propose through this article that is the analysis of the approach presented in[2] by putting it in a setting of mixed formulation where the measure μ appears as a Lagrange multiplier. It allows us to adapt the theory of Brezzi[6] and[7] for the analysis of the approximation by finite element of the reformulated problem.

This point of view is different from the one presented in[2] since one determines the two unknowns of the obstacle problem at the same time, u and μ (and so the free boundary), and also different from the one adopted by the different mixed formulations of the obstacle problem.

2. Survey of the abstract problem

2.1. *The continuous problem*

Let us consider two Hilbert spaces H and V such that $V \subset H \subset V'$, with continuous and dense injections.

We denote $\langle ., . \rangle$ the duality pairing between V and V', $a(., .)$ the scalar product of V and $(., .)$ the one of H. And we will note A the Riesz operator between V and V' (i.e. $\forall (u, v) \in V^2 : a(u, v) = \langle Au, v \rangle$).

We consider a closed convex cone K in H, its polar cone is defined by:

$$K^0 = \{ w \in H \text{ such that } \forall v \in K : (v, w) \leq 0 \}.$$

If $v \in H$, v^+ and v^- designate respectively the orthogonal projections of v on K and K^0, i.e.

$$v = v^+ + v^-, (v^+, v^-) = 0 \text{ and } \forall w \in K : (v - v^+, w - v^+) \leq 0.$$

We suppose in addition that:

(H1) K is right i.e. $K^0 = -K$.
(H2) $\forall v \in V : v^+, v^- \in V$ and $a(v^+, v^-) = 0$.
(H3) $\forall u, v \in V : u^+ + v^+ - (u+v)^+ \in K \cap V$.

We consider the following variational inequality problem:

$$\begin{cases} \text{Find } u \in K \cap V \quad \text{such that} \\ a(u, v-u) + (f, v-u) \geq 0, \text{ for every } v \in K \cap V, \end{cases} \tag{4}$$

where $f \in H$. It is known that problem (4) admits a unique solution.[1]

2.2. *Reformulation of the problem*

As in[2] we consider, on V, the continuous and convex mapping

$$\varphi : V \longrightarrow \mathbb{R},\ v \longmapsto \varphi(v) = (f^+, v^+).$$

The following propositions have been shown in.[2]

Proposition 2.1. *Problem (4) is equivalent to the problem*

$$\begin{cases} \textit{Find } u \in V \ \textit{ such that} \\ a(u, v-u) + \varphi(v) - \varphi(u) + (f^-, v-u) \geq 0, \textit{ for every } v \in V. \end{cases} \tag{5}$$

Problem (5) is different from (4) by the fact that it is without constraints. In addition, formulation (5) allows us to show the following continuity result.

Lemma 2.1. *if u is the solution to problem (5) then the linear form,*

$$w \longmapsto a(u, w),$$

is continuous on V for the norm of H. And one has the bound of the norm of u in H :

$$\sup_{w \in V, w \neq 0} \frac{a(u,w)}{\|w\|_H} \leq \|f^+\|_H + \|f^-\|_H \,. \tag{6}$$

Proof. For every w in V, with $v = u + w$ in (5) and thanks to (H3), one has:

$$\begin{aligned} a(u,w) &\geq \left(f^+, (u^+ + w^+) - (u+w)^+\right) - \left(f^+, w^+\right) - (f^-, w) \\ &\geq -\left(f^+, w^+\right) - (f^-, w) \\ &\geq -\left\{\left\|f^+\right\|_H + \left\|f^-\right\|_H\right\} \|w\|_H . \end{aligned}$$

And with $v = u - w$, one has:

$$a(u, -w) \geq -\left\{\left\|f^+\right\|_H + \left\|f^-\right\|_H\right\} \|-w\|_H ,$$

that is to say

$$a(u, w) \leq \left\{\left\|f^+\right\|_H + \left\|f^-\right\|_H\right\} \|w\|_H .$$

That finishes the proof. □

Proposition 2.2. *The function u is the solution to problem (5) if and only if (u, μ) is the solution to problem*

$$\begin{cases} \textit{Find } (u, \mu) \in V \times H \textit{ such that} \\ a(u,v) + (\mu, v) + (f^-, v) = 0, \textit{ for every } v \in V, \\ \mu \in \partial\varphi(u), \end{cases} \tag{7}$$

$\partial\varphi(u) = \{\lambda \in H \mid \forall w \in H : \varphi(u) - \varphi(w) \leq (\lambda, u - w)\}$ *designates the subdifferential of φ at u.*

Taking into account the following lemma 2.2 (Lemma 1 in[2]), problem (7) can be written as:

$$\begin{cases} \text{Find } (u, \mu) \in V \times C \text{ such that} \\ a(u,v) + \langle \mu, v \rangle + (f^-, v) = 0, \text{ for every } v \in V, \\ \langle \zeta - \mu, u \rangle \leq 0, \text{ for every } \zeta \in C, \end{cases} \tag{8}$$

where $C = \{\zeta \in H : \langle \zeta, v \rangle \leq \varphi(v), \text{ for every } v \in V\}$ is a closed convex containing 0_H.

Lemma 2.2. *For every $\mu \in H$ the following propositions are equivalent*

(i) $\mu \in \partial\varphi(u)$.
(ii) $\mu \in C$ *and* $\langle \mu, u \rangle = \varphi(u)$.
(iii) $\mu \in C$ *and* $\langle \zeta - \mu, u \rangle \leq 0$ *for every* $\zeta \in C$.

In,[2] for the calculation of μ then the resolution of a Dirichlet problem to determine u, the authors proposed to uncouple problem (8) by putting $z = A^{-1}(\mu)$ and $t = A^{-1}(f^-)$. Problem (8) is then written:

$$\begin{cases} \text{Find } (u, z) \in V \times M \text{ such that} \\ a(u + z + t, v) = 0, \text{ for every } v \in V, \\ a(w - z, u) \leq 0, \text{ for every } w \in M, \end{cases}$$

where A is the Riesz-Fréchet operator associated to $a(.,.)$ and $M = A^{-1}(C)$. That permits to define z as being the projection of $-t$ on the closed convex M for the scalar product $a(.,.)$. And in order to compute z, they proposed a projection algorithm inspired from the one of Degueil.[8]

The set M being defined only implicitly, so the proposed algorithm cannot be implemented, and makes the analysis of the discrete problem nearly impossible. In addition to calculate t it is necessary to solve the problem $At = f^-$ in advance.

What we propose, in alternative, it is the direct resolution (through an approximation by finite element) of **the mixed problem** (8), especially as the convex C can be determined explicitly, more precisely one has.

Lemma 2.3. *With the hypotheses above one has*

$$C = \left(f^+ + K^0\right) \cap K.$$

Proof. By definition of C and φ, one has, for every $\zeta \in C$

$$\forall v \in V : (\zeta, v) \leq (f^+, v^+).$$

The space V being dense in H and the projection operator from H on the cone K is continuous, so the previous inequality is verified for all v in H.

While taking v respectively in K and K^0, one gets

$$\forall v \in K : \left(\zeta - f^+, v\right) \leq 0 \text{ and } \forall v \in K^0 : (\zeta, v) \leq (f^+, v^+) = 0,$$

then $\zeta - f^+ \in K^0$ and $\zeta \in \left(K^0\right)^0 = K$.

On the other hand if $\zeta \in K$ and $\zeta - f^+ \in K^0$, one has, for all $v \in V$

$$(\zeta, v) = \left(\zeta, v^+\right) + \left(\zeta, v^-\right) \leq \left(\zeta, v^+\right) \leq (f^+, v^+),$$

so $\zeta \in C$. □

Remark 2.1. While the convex C is not a cone, one cannot apply directly the theory of Brezzi.[6] In this case an alternative is proposed in[7] to justify existence and uniqueness of the solution to problem (8) through an inf-sup condition between the spaces V and H equipped with the norm of V', which is obvious in our case, indeed, for every $\alpha \in V'$ one has:

$$\sup_{v \in V, v \neq 0} \frac{\langle \alpha, v \rangle}{\|v\|_V} = \|\alpha\|_{V'} . \tag{9}$$

Remark 2.2. The bilinear form a being symmetric, so the mixed problem (8) is equivalent to the following saddle point problem:[9]

$$\begin{cases} \text{Find } (u, \mu) \in V \times C \text{ such that} \\ \mathcal{L}(u, \zeta) \leq \mathcal{L}(u, \mu) \leq \mathcal{L}(v, \mu) \text{ , for all } (v, \zeta) \in V \times C, \end{cases} \tag{10}$$

where the Lagrangian $\mathcal{L}$ is defined on $V \times V'$ by $\mathcal{L}(v, \zeta) = \dfrac{1}{2} a(v, v) + (f^-, v) + \langle \zeta, v \rangle$.

2.3. *Property of stability*

Seen should be given the inf-sup condition (9) problem (8) can be studied regardless of problem (4) (see[7]), so we have the following stability result.

Proposition 2.3. *There exists a positive constant C such that*

$$\|u\|_V + \|\mu\|_{V'} \leq C \left\| f^- \right\|_H , \tag{11}$$

where (u, μ) denote the solution of problem (8), and $\|.\|_X$ the norm of the Hilbert space X.

Proof. With $v = -u$ and $\zeta = 0$ in (8), one has

$$a(u, -u) + \langle -\mu, u \rangle + (f^-, u) = 0 \text{ and } \langle -\mu, u \rangle \leq 0,$$

hence

$$\|u\|_V \leq c \left\| f^- \right\|_H . \tag{12}$$

To estimate $\|\mu\|_{V'}$, we also use (8), so for all v in V one has:

$$\begin{aligned} \langle \mu, v \rangle &= -(f^-, v) - a(u, v) \\ &\leq \left\| f^- \right\|_H \|v\|_H + \|u\|_V \|v\|_V \\ &\leq C \left\{ \left\| f^- \right\|_H + \|u\|_V \right\} \|v\|_V , \end{aligned}$$

that implies with (12) the estimation (11). □

Remark 2.3. The stability result above shows how much f^- controls the solution of the problem. in particular if $f^- = 0$ then $u = \mu = 0$.

Remark 2.4. If the constraint of problem (4) is given by the translated $w+K$ of K by an element $w \in W$ (with $V \underset{\rightarrow}{\subseteq} W \underset{\rightarrow}{\subseteq} H$ such that the bilinear form a is defined and continuous on W), problem (5) can be written

$$\begin{cases} \text{Find } u \in V \text{ such that} \\ a(u-w, v-u) + \varphi_1(v) - \varphi_1(u) + (F^-, v-u) \geq 0, \text{ for all } v \in V, \end{cases} \tag{13}$$

where $\varphi_1 : v \longmapsto (F^+, (v-w)^+)$ and F defined by $(F, v) = (f, v) + a(w, v)$ for all $v \in V$, we suppose that F is continuous for the norm of H. Problem (8) becomes:

$$\begin{cases} \text{Find } (u, \mu) \in V \times C' \text{ such that} \\ a(u-w, v) + \langle \mu, v \rangle + (F^-, v) = 0, \text{ for every } v \in V, \\ \langle \zeta - \mu, u - w \rangle \leq 0, \text{ for every } \zeta \in C', \end{cases} \tag{14}$$

where $C' = \left(F^+ + K^0\right) \cap K$.

2.4. *The discrete problem*

Let $(V_h)_h$ be a sequence of finite dimensional subspaces of V, and for all h we define V_h' as the dual of V_h by

$$V_h' = \{a(v_h, .) \text{ such that } v_h \in V_h\} = \{(v_h, .) \text{ such that } v_h \in V_h\} \subset V',$$

subspace of V' and of H (V_h being of finite dimension so the restrictions to V_h of the two norms $\|.\|_V$ and $\|.\|_H$ are equivalent).

We consider, finally, a sequence (C_h) of closed convex subsets of V_h containing $0_{V'}$. And we assume that

$$\text{the sequence } (V_h, C_h) \text{ approximates } (V, C) \text{ in the sense of.}^5 \tag{15}$$

The discrete problem associated to (8) can be written as:

$$\begin{cases} \text{Find } (u_h, \mu_h) \in V_h \times C_h \text{ such that} \\ a(u_h, v_h) + \langle \mu_h, v_h \rangle + (f^-, v_h) = 0, \text{ for all } v_h \in V_h, \\ \langle \zeta_h - \mu_h, u_h \rangle \leq 0, \text{ for all } \zeta_h \in C_h. \end{cases} \tag{16}$$

As in,[7] we introduce the following discrete inf-sup condition:

$$\forall \alpha_h \in V_h' : \sup_{v_h \in V_h, v_h \neq 0} \frac{\langle \alpha_h, v_h \rangle}{\|v_h\|_V} \geq \beta \|\alpha_h\|_{V'}, \tag{17}$$

for some $\beta > 0$ independent of h. What permits to show (see[7]) the following stability result.

Proposition 2.4. *If the assumption (17) holds then there exists a constant c such that*

$$\|u_h\|_V + \|\mu_h\|_{V'} \leq c \left\|f^-\right\|_H ,$$

where (u_h, μ_h) *denotes the solution of problem (16).*

In order to show a convergence result, first one has to show the following estimate, necessary especially for the error estimate. Proofs can be found in[7] and.[10]

Theorem 2.1. *Under assumption (17), if* (u, μ) *and* (u_h, μ_h) *denote the respective solutions to problems (8) and (16), then there exists a constant c independent of h such that the following estimates hold:*

$$\|u - u_h\|_V^2 \leq c \Big\{ \inf_{v_h \in V_h} \|u - v_h\|_V^2 + \inf_{\zeta_h \in C_h} \Big(\|\mu - \zeta_h\|_{V'}^2 + B_1(\zeta_h) \Big) + \inf_{\zeta \in C} B_2(\zeta) \Big\}, \tag{18}$$

$$\|\mu - \mu_h\|_{V'}^2 \leq c \left\{ \|u - u_h\|_V^2 + \inf_{\zeta_h \in C_h} \|\mu - \zeta_h\|_{V'}^2 \right\}, \tag{19}$$

where $B_1(\zeta_h) = \langle \mu - \zeta_h, u \rangle$ *and* $B_2(\zeta) = \langle \mu_h - \zeta, u \rangle$.

We finally state the convergence result.

Theorem 2.2. *Under assumptions (15) and (17), if* (u, μ) *and* (u_h, μ_h) *denote the respective solutions to problems (8) and (16), then* (u_h, μ_h) *converges strongly to* (u, μ) *in* $V \times V'$.

3. Study of an obstacle problem

3.1. *The continuous problem*

Let Ω be a bounded domain (that one supposes polygonal to simplify the analysis) in $\mathbb{R}^n$ $(n \leq 2)$. The Hilbert space $H_0^1(\Omega)$ is equipped with the scalar product:

$$a(u, v) = \int_\Omega \nabla u \nabla v dx, \text{ for all } (u, v) \in H_0^1(\Omega).$$

Let $f \in H^{-1}(\Omega)$ and $\psi \in H^1(\Omega)$ such that $\psi \leq 0$ on $\partial\Omega$, we suppose that

$$F = (f - \Delta\psi) \in L^2(\Omega),$$

and we define

$$K_\psi = \left\{v \in L^2(\Omega) : v \geq \psi \text{ a.e. in } \Omega\right\}.$$

We consider the following obstacle problem:

$$\begin{cases} \text{Find } u \in K_\psi \cap H_0^1(\Omega) \text{ such that} \\ a(u, v-u) + (f, v-u) \geq 0, \text{ for all } v \in K_\psi \cap H_0^1(\Omega). \end{cases} \tag{20}$$

The hypotheses of the section (2) are verified for $V = H_0^1(\Omega)$, $W = H^1(\Omega)$, $H = L^2(\Omega)$, and $K_\psi = \psi + K$ $(w = \psi)$, with K the convex cone of $L^2(\Omega)$ of almost everywhere positive functions in Ω i.e.

$$K = -K^0 = \left\{v \in L^2(\Omega) : v \geq 0 \text{ a.e. in } \Omega\right\}.$$

Problem (20) is then equivalent to the following one:

$$\begin{cases} \text{Find } (u, \mu) \in H_0^1(\Omega) \times C' \text{ such that} \\ a(u-\psi, v) + (\mu, v) + (F^-, v) = 0, \text{ for all } v \in H_0^1(\Omega), \\ (\zeta - \mu, u - \psi) \leq 0, \text{ for all } \zeta \in C', \end{cases} \tag{21}$$

with $C' = K \cap \left(F^+ + K^0\right) = \left\{\zeta \in L^2(\Omega) \text{ such that } 0 \leq \zeta \leq F^+ \text{ a.e. in } \Omega\right\}$.

On the other hand if (u, μ) is solution to problem (21) then u is the solution to the following Dirichlet problem:

$$\begin{cases} \Delta u = \mu + F^- + \Delta\psi \text{ in } \Omega, \\ u_{|\partial\Omega} = 0. \end{cases} \tag{22}$$

In addition it is known[1] that the coincidence set $[u = \psi]$ is contained in the set $[F \geq 0]$ and

$$\Delta(u - \psi) = F\chi_{[u>\psi]}, \tag{23}$$

what means:

$$F^-\chi_{[u=\psi]} = 0, \tag{24}$$

and

$$\Delta u = F^+\chi_{[u>\psi]} + F^-\{1 - \chi_{[u=\psi]}\} + \Delta\psi = F^+\chi_{[u>\psi]} + F^- + \Delta\psi \text{ in } \Omega. \tag{25}$$

Combining (22) and (25) we can write

$$\mu = F^+\chi_{[u>\psi]} \text{ a.e in } \Omega. \tag{26}$$

Remark 3.1. In order to simplify the analysis, we will suppose that $\psi = 0$. Seen should be given the form of problem (21), it is clear that this hypothesis is not restrictive.

3.2. *The discrete problem*

Let us consider that a triangulation $\mathcal{T}_h$ is defined over Ω, regular in the sense that each simplex $T \in \mathcal{T}_h$ contains a ball with radius $\gamma_1 h$ and is contained in a ball with radius $\gamma_2 h$ where the positive constants γ_1 and γ_2 are independent of h.

A piecewise linear subspace V_h of V can be defined as:

$$V_h = \left\{ v_h \in \mathcal{C}^0\left(\bar{\Omega}\right) \cap H_0^1(\Omega) \text{ such that for all } T \in \mathcal{T}_h : v_{h|T} \in \mathcal{P}_1(T) \right\}.$$

Let $N = N(h)$, number of nodes A_i of the triangulation, to simplify the notations, we assume that the numbering is such that $A_1, A_2, \ldots, A_{N_0}$ are the internal nodes while $A_{N_0+1}, \ldots, A_N$ lie on the boundary Γ. N_0 is then the dimension of V_h. We introduce the canonical basis $(\varphi_1, \varphi_2, \ldots, \varphi_N)$ associated to the triangulation $\mathcal{T}_h$.

We assume in addition that the inverse assumption (see[11]) holds, what permits to have the following inverse inequality:

$$\|w_h\|_{H_0^1(\Omega)} \leq ch^{-1} \|w_h\|_{L^2(\Omega)}, \forall w_h \in V_h. \tag{27}$$

To approximate $\mu \in L^2(\Omega)$, we need a discrete space with the same dimension as V_h, so we can use V_h or we associate to the triangulation $\mathcal{T}_h$ the set $\mathcal{K}_h$ of the N volumes D_i $(i = 1...N)$ that constitute the dual of the triangulation $\mathcal{T}_h$ known as the Voronoi mesh. This mesh is constructed by connecting with a straight line segment the mid-points of edges and centroids of each neighboring pair of triangles having a common edge. And we introduce the space W_h (approximation of $L^2(\Omega)$ and $H^{-1}(\Omega)$), defined by:

$$W_h = \left\{ w_h \in L^2\left(\Omega\right) \text{ such that for all } D \in \mathcal{K}_h : w_{h|D} \in \mathcal{P}_0(D) \right\},$$

equipped with its canonical basis $(\chi_1, \chi_2, ..., \chi_{N_0})$, where χ_i is the characteristic function of the control volume D_i.

The discrete problem associated to (21) reads:

$$\begin{cases} \text{Find } (u_h, \mu_h) \in V_h \times C_h \text{ such that} \\ a(u_h, v_h) + \langle \mu_h, v_h \rangle + (f^-, v_h) = 0, \text{ for all } v_h \in V_h, \\ \langle \zeta_h - \mu_h, u_h \rangle \leq 0, \text{ for all } \zeta_h \in C_h, \end{cases} \tag{28}$$

where we set

$$C_h = \left\{ \zeta_h = \sum_{i=1}^{N_0} \zeta_i \chi_i \in W_h \text{ such that } 0 \leq \langle \zeta_h, \varphi_i \rangle \leq T_i^+ : i = 1 \ldots N_0 \right\},$$

and $T_i^+ = \int_\Omega f^+ \varphi_i dx$.

With the notations above it is easy to see that problem (28) can be written as:

$$\begin{cases} \text{Find } (U,Y) \in \mathbb{R}^{N_0} \times \prod_{i=1}^{N_0} \left[0, T_i^+\right] \text{ such that} \\ \mathcal{J}(U,z) \leq \mathcal{J}(U,Y) \leq \mathcal{J}(w,Y), \text{ for all } (w,z) \in \mathbb{R}^{N_0} \times \prod_{i=1}^{N_0} \left[0, T_i^+\right], \end{cases} \tag{29}$$

where we set $\mathcal{J}(w,z) = \dfrac{1}{2}(Mw,w) + (T^-,w) + (z,w)$ with $T_i^- = \int_\Omega f^- \varphi_i dx$ and $Y = P\mu_h$. M is the stiffness matrix of the problem i.e. $M = (m_{ij} = a(\varphi_i, \varphi_j))$, $P = (p_{ij})$, is the $N_0 \times N_0$-matrix such that $p_{ij} = \displaystyle\int_\Omega \varphi_i \chi_j dx = \int_{D_j} \varphi_i dx$. In the one dimensional case and with regular triangulation, the matrix P is easily calculable and one has:

$$P = \frac{h}{8} \begin{pmatrix} 6 & 1 & 0 & \cdots & 0 \\ 1 & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & 1 & 6 \end{pmatrix}.$$

In the two dimensional case the matrix P is also calculable for a triangulation with simple geometries and it is symmetric. More precisely one can show, in an elementary manner, the following properties: $p_{ii} = \dfrac{11}{18}|D_i|$ and $\displaystyle\sum_{j \neq i} p_{ij} = \frac{7}{18}|D_i|$ for $i = 1...N_0$. To compute $p_{ij} = \displaystyle\int_{D_j} \phi_i dx$ for $i = j$ or for internal neighboring nodes A_i and A_j, one can use trapezoid formula by splitting D_j into small triangles and using the fact that the dual mesh splits any triangle of the primal triangulation into six small triangles having common measure.

If one uses V_h instead of W_h to approach μ, the matrix P will be replaced by $B = (b_{ij})$, where $b_{ij} = \displaystyle\int_\Omega \varphi_i \varphi_j dx$. Using Simpson's formula, one can show the following properties: $b_{ii} = \dfrac{1}{3}|K_i|$ and $\displaystyle\sum_{j \neq i} b_{ij} = \frac{1}{6}|K_i|$, where $K_i = \displaystyle\bigcup_{A_i \in T} T$, for all internal node A_i, $i = 1...N_0$.

To use the results of the section (2) we must establish a discrete inf-sup condition and show that C_h approximates C in the sense of.[5] To this end, we establish the following technical lemmas.

Lemma 3.1. *There exists two positive constants c_1, c_2 independent of h such that*

$$c_1 \|w_h\|^2_{L^2(\Omega)} \leq \|\alpha_h\|^2_{L^2(\Omega)} \leq c_2 \langle \alpha_h, w_h \rangle , \tag{30}$$

for all $\alpha_h = \sum_i \alpha_i \chi_i \in W_h$, where $w_h = \sum_i \alpha_i \varphi_i \in V_h$.

Proof. In 2-D case.

For $\alpha = {}^t(\alpha_1, ..., \alpha_N)$, $\alpha_h = \sum_i \alpha_i \chi_i \in W_h$ and $w_h = \sum_i \alpha_i \varphi_i \in V_h$, one has

$$\|\alpha_h\|^2_{L^2(\Omega)} = {}^t\alpha D\alpha, \qquad \|w_h\|^2_{L^2(\Omega)} = {}^t\alpha B\alpha, \text{ and } \langle \alpha_h, w_h \rangle = {}^t\alpha P\alpha,$$

where D is the diagonal matrix $D = diag(|D_1|, ..., |D_N|)$. So

$$\frac{\|w_h\|^2_{L^2(\Omega)}}{\|\alpha_h\|^2_{L^2(\Omega)}} = \frac{{}^t\alpha B\alpha}{{}^t\alpha D\alpha} = \frac{{}^t\alpha B\alpha}{{}^t\alpha\alpha}\frac{{}^t\alpha\alpha}{{}^t\alpha D\alpha} \leq \frac{t}{s}, \tag{31}$$

where t is the largest eigenvalue of B, and s the smallest one of D. By the regularity assumptions on the triangulation and the properties of B and D, and thanks to Gershgorin's theorem, one has

$$t \leq \frac{1}{2} \max_i |K_i| \leq ch^2 \text{ and } s \geq \min_i |D_i| \geq ch^2,$$

the first inequality of (30) follows.

On the other hand

$$\frac{\|\alpha_h\|^2_{L^2(\Omega)}}{\langle \alpha_h, w_h \rangle} = \frac{{}^t\alpha D\alpha}{{}^t\alpha P\alpha} = \frac{{}^t\alpha D\alpha}{{}^t\alpha\alpha}\frac{{}^t\alpha\alpha}{{}^t\alpha P\alpha} \leq \frac{t'}{s'}, \tag{32}$$

where t' is the largest eigenvalue of D, and s' the smallest one of P. While reasoning like previously, the second inequality of (30) follows. □

Lemma 3.2. *For $\lambda \in L^2(\Omega)$, if λ^I is the projection of λ on W_h in the following sense*

$$\int_\Omega \lambda v_h dx = \int_\Omega \lambda^I v_h dx, \forall v_h \in V_h,$$

then there exists a positive constant c independent of h and λ such that

$$\left\|\lambda - \lambda^I\right\|_{H^{-1}(\Omega)} \leq ch \|\lambda\|_{L^2(\Omega)} . \tag{33}$$

Proof. For all $v \in H_0^1(\Omega)$, if $v^I \in V_h$ denotes the Lagrange interpolant of v, one has

$$\begin{aligned}\int_\Omega (\lambda - \lambda^I) v dx &= \int_\Omega (\lambda - \lambda^I)(v - v^I) dx \\ &\leq \left\|\lambda - \lambda^I\right\|_{L^2(\Omega)} \left\|v - v^I\right\|_{L^2(\Omega)} \\ &\leq ch \left\|\lambda - \lambda^I\right\|_{L^2(\Omega)} \|v\|_{H_0^1(\Omega)},\end{aligned}$$

therefore

$$\left\|\lambda - \lambda^I\right\|_{H^{-1}(\Omega)} \leq ch \left\|\lambda - \lambda^I\right\|_{L^2(\Omega)}. \tag{34}$$

On the other hand, let $\lambda^I = \sum_i \lambda_i \chi_i$, and $w_h = \sum_i \lambda_i \varphi_i$, using (30), one has

$$\left\|\lambda^I\right\|_{L^2(\Omega)}^2 \leq c \int_\Omega \lambda^I w_h dx = c \int_\Omega \lambda w_h dx \leq c \|\lambda\|_{L^2(\Omega)} \left\|\lambda^I\right\|_{L^2(\Omega)},$$

what implies, with the help of (34), the estimate (33). □

Lemma 3.3. *There exists a constant β independent of h, such that*

$$\forall \alpha_h \in W_h : \sup_{v_h \in V_h, v_h \neq 0} \frac{\langle \alpha_h, v_h \rangle}{\|v_h\|_{H_0^1(\Omega)}} \geq \beta \|\alpha_h\|_{H^{-1}(\Omega)}. \tag{35}$$

Proof. For $\alpha_h = \sum_{i=1}^N \alpha_i \chi_i \in W_h$, let $w \in H_0^1(\Omega) \cap H^2(\Omega)$ and $w_h \in V_h$, such that

$$-\Delta w = \alpha_h \text{ and } a(w_h, v_h) = a(w, v_h) = \langle \alpha_h, v_h \rangle, \forall v_h \in V_h,$$

it is known (see[12]), that $\|w - w_h\|_{H_0^1(\Omega)} \leq ch \|\alpha_h\|_{L^2(\Omega)}$, therefore

$$\|w\|_{H_0^1(\Omega)}^2 - \|w_h\|_{H_0^1(\Omega)}^2 = \|w - w_h\|_{H_0^1(\Omega)}^2 \leq ch^2 \|\alpha_h\|_{L^2(\Omega)}^2,$$

hence

$$\|\alpha_h\|_{H^{-1}(\Omega)}^2 = \|w\|_{H_0^1(\Omega)}^2 \leq \|w_h\|_{H_0^1(\Omega)}^2 + ch^2 \|\alpha_h\|_{L^2(\Omega)}^2. \tag{36}$$

On the other hand, for $v_h = \sum_{i=1}^N \alpha_i \varphi_i \in V_h$, using (30) and the inverse inequality (27), one has

$$\frac{\langle \alpha_h, v_h \rangle}{\|v_h\|_{H_0^1(\Omega)}} \geq ch \|\alpha_h\|_{L^2(\Omega)}, \tag{37}$$

what permits to conclude, with the help of (36) that

$$\|\alpha_h\|_{H^{-1}(\Omega)} \leq c \|w_h\|_{H_0^1(\Omega)} = c \sup_{v_h \in V_h, v_h \neq 0} \frac{\langle \alpha_h, v_h \rangle}{\|v_h\|_{H_0^1(\Omega)}}.$$

If one uses V_h instead of W_h to approximate μ, the discrete inf-sup condition would be an immediate consequence of (36) and (27). □

Lemma 3.4. *The sequence (C_h) approximates C in the following sense:*

(i) $\forall \lambda \in C, \exists \lambda_h \in C_h$,*such that:* $\lambda_h \to \lambda$ *in* $H^{-1}(\Omega)$.
(ii) If $\lambda_h \in C_h$ *such that* $\lambda_h \rightharpoonup \lambda$ *in* $H^{-1}(\Omega)$, *for some* λ *in* $L^2(\Omega)$, *then* $\lambda \in C$.

Proof. Item (i) follows from the estimate (33) and the fact that $\lambda^I \in C_h$ if $\lambda \in C$.

On the other hand let λ_h and λ as in (ii), so for all $v \in H_0^1(\Omega)$, $v \geq 0$ and $v_h \in V_h$, $v_h \geq 0$ such that $v_h \to v$ in $H_0^1(\Omega)$, one has

$$0 \leq \int_\Omega \lambda_h v_h dx \leq \int_\Omega f^+ v_h dx,$$

however $\int_\Omega \lambda_h v_h dx \to \int_\Omega \lambda v dx$, therefore

$$0 \leq \int_\Omega \lambda v dx \leq \int_\Omega f^+ v dx,$$

what permits to conclude that $\lambda \in C$. □

In addition to the results of Section 2, one has the following error estimate.

Theorem 3.1. *If (u, μ) and (u_h, μ_h) designate the respective solutions to problems (21) and (28), and if $u \in H^2(\Omega)$ then there exists a positive constant c independent of h such that:*

$$\|u - u_h\|_{H_0^1(\Omega)} + \|\mu - \mu_h\|_{H^{-1}(\Omega)} \leq ch. \tag{38}$$

Proof. Using theorem 2.1, and thanks to the interpolation error (see[12]):

$$\left\|u - u^I\right\|_{L^2(\Omega)} \leq ch^2 \|u\|_{H^2(\Omega)} \text{ and } \left\|u - u^I\right\|_{H_0^1(\Omega)} \leq ch \|u\|_{H^2(\Omega)},$$

where $u^I \in V_h$ denotes the interpolant of $u \in H_0^1(\Omega)$, and the interpolation (33):

$$\left\|\mu - \mu^I\right\|_{H^{-1}(\Omega)} \leq ch \|\mu\|_{L^2(\Omega)},$$

we had proved, it rests to examine the quantities $B_1(\zeta_h)$ and $B_2(\zeta)$ that appear in the inequalities (18) and (19). To this end, let us take $\zeta_h = \mu^I \in C_h$ in B_1, we have:

$$\begin{aligned} B_1(\mu^I) &= \left\langle \mu - \mu^I, u \right\rangle = \left\langle \mu - \mu^I, u - u^I \right\rangle \\ &\leq \left\| \mu - \mu^I \right\|_{H^{-1}(\Omega)} \left\| u - u^I \right\|_{H_0^1(\Omega)} \\ &\leq ch^2 \left\| \mu \right\|_{L^2(\Omega)} \left\| u \right\|_{H^2(\Omega)} . \end{aligned}$$

On the other hand, with $\zeta = \mu \in C$ in B_2, and while taking into account the fact that $u \geq 0$ a.e. in Ω and

$$\langle \mu, u \rangle = \left\langle f^+, u \right\rangle ,$$

we obtain:

$$\begin{aligned} B_2(\mu) &= \langle \mu_h - \mu, u \rangle = \langle \mu_h - f^+, u - u^I \rangle + \langle \mu_h - f^+, u^I \rangle \\ &\leq \langle \mu_h - f^+, u - u^I \rangle \text{ (because } \mu_h \in C_h \text{ and } u^I \geq 0) \\ &\leq \left\| \mu_h - f^+ \right\|_{L^2(\Omega)} \left\| u - u^I \right\|_{L^2(\Omega)} \\ &\leq \left\| \mu_h - f^+ \right\|_{L^2(\Omega)} ch^2 \left\| u \right\|_{H^2(\Omega)} , \end{aligned}$$

to conclude, we must show that $\|\mu_h\|_{L^2(\Omega)}$ is bounded independently of h, more precisely we show that

$$\|\mu_h\|_{L^2(\Omega)} \leq c \left\| f^+ \right\|_{L^2(\Omega)} , \tag{39}$$

indeed, let $\mu_h = \sum_{i=1}^{N_0} \mu_i \chi_i$ and $v_h = \sum_{i=1}^{N_0} \mu_i \varphi_i$, one has (lemma 3.1):

$$\begin{aligned} \|\mu_h\|^2_{L^2(\Omega)} &\leq c_2 \langle \mu_h, v_h \rangle = c_2 \sum_{i=1}^{N_0} \mu_i \int_\Omega \mu_h \varphi_i dx \\ &\leq c_2 \sum_{i=1}^{N_0} |\mu_i| \int_\Omega \mu_h \varphi_i dx \leq c_2 \sum_{i=1}^{N_0} |\mu_i| \int_\Omega f^+ \varphi_i dx \\ &\leq c_2 \int_\Omega f^+ \sum_{i=1}^{N_0} |\mu_i| \, \varphi_i dx \leq c_2 \left\| f^+ \right\|_{L^2(\Omega)} \left\| \sum_{i=1}^{N_0} |\mu_i| \, \varphi_i \right\|_{L^2(\Omega)} \\ &\leq c \left\| f^+ \right\|_{L^2(\Omega)} \left\| \sum_{i=1}^{N_0} |\mu_i| \, \chi_i \right\|_{L^2(\Omega)} \\ &\leq c \left\| f^+ \right\|_{L^2(\Omega)} \|\mu_h\|_{L^2(\Omega)} . \end{aligned}$$

That finishes the proof.

In the case where one chooses V_h instead of W_h to approximate μ , the inequality (39) can be proved in the same manner by putting $\mu_h = \sum_{i=1}^{N_0} \mu_i \varphi_i$, and using two times the lemma 3.1. □

3.3. *Some numerical results*

In this section, we present two numerical tests in order to confirm the theoretical results.

Example 1.

We consider the following one-dimensional obstacle problem, with $\Omega =]0, 2[$, $\psi = 0$ and f defined by:

$$\begin{cases} f(x) = -1 \text{ if } x \in [0, 1]\,, \\ f(x) = 1 \text{ if } x \in]1, 2]\,. \end{cases}$$

The solution to the continuous problem is calculable. And one has:

$$\begin{cases} u(x) = -\frac{1}{2}x^2 + (2 - \sqrt{2})x \text{ if } x \in [0, 1]\ , \\ u(x) = \frac{1}{2}x^2 - \sqrt{2}x + 1 \text{ if } x \in \left]1, \sqrt{2}\right]\,, \\ u(x) = 0 \text{ if } x \in \left]\sqrt{2}, 2\right]\,, \end{cases}$$

and the measure $\mu = \chi_{[1,\sqrt{2}]}$.

We solve the mixed problem (29) using Uzawa's algorithm, and for different values of h. Figure 1 bellow represents the L^2-error on μ for different values of h.

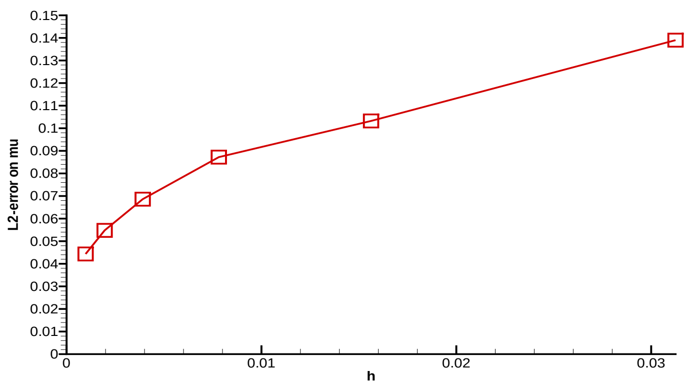


Fig. 1. L^2-error on μ for different values of h.

Figure 2 gives a comparison between the curve of u and the one of u_h for $h = 1/32$. One remarks that the two curves are nearly confounded, and this even though h is not small enough.

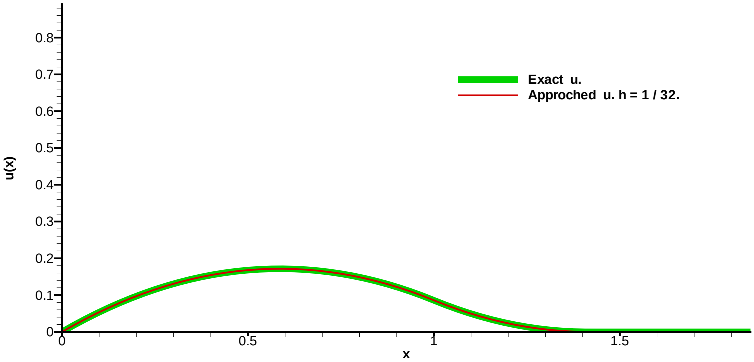


Fig. 2. Comparison between u and u_h for $h = 1/32$.

Figure 3 gives a comparison between the curve of μ and the one of μ_h for $h = 1/256$. The oscillations near the value 1 are due to the discontinuity of f at this point.

The advantage to use μ to characterize the free boundary and not u, is due to the fact that with $\mu = F^+\chi_{[u>\psi]}$ the jump is pronounced (because under some regularity hypotheses on the obstacle ψ (see[13]) one has $F \geq c > 0$ near the free boundary), what is not the case for u that leaves the obstacle with a speed at least quadratic. It is what one observes well on the two figures 2 and 3.

Example 2.

We consider in the 2-D case, the following problem, with $\Omega =]-1,1[^2$, $\psi = 0$ and f as:

$$f(x,y) = 8x^2 + 8y^2 - 1.$$

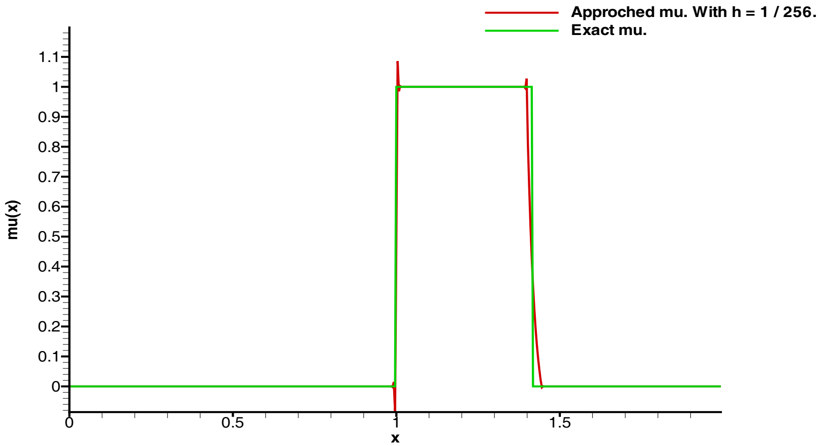


Fig. 3. Comparison between μ and μ_h for $h = 1/256$.

The solution to the obstacle problem is the function u defined on Ω by:

$$\begin{cases} u(x,y) = \dfrac{1}{32}(4x^2+4y^2-1)^2 \quad \text{if} \quad 4x^2+4y^2-1 \leq 0, \\ u(x,y) = 0 \quad \text{if} \quad 4x^2+4y^2-1 \geq 0. \end{cases}$$

We solve the mixed problem for different values of h, results obtained seem to be in agreement with the established theoretical results.

Figures 4 and 5 represent respectively the isovalues of the functions u and u_h for $h = 1/30$. We can easily notice the strong similarities between the two figures.

In the same manner Figures 6 and 7 represent respectively the isovalues of exact μ and μ_h, the approximation of μ. for $h = 1/60$, we notice also the similarities of the two figures.

Figure 8 represents the $H_0^1(\Omega)$-error on u for different values of h, one can notice the linear aspect of this evolution, what confirms the error estimate of the theorem 3.1.

Finally, we note that 2-D simulations have been realized under the environment *FreeFem++* .[14]

Current and future developments

The aim of this work was to present a new mixed formulation of the obstacle problem. We have proved that this formulation is as effective as those

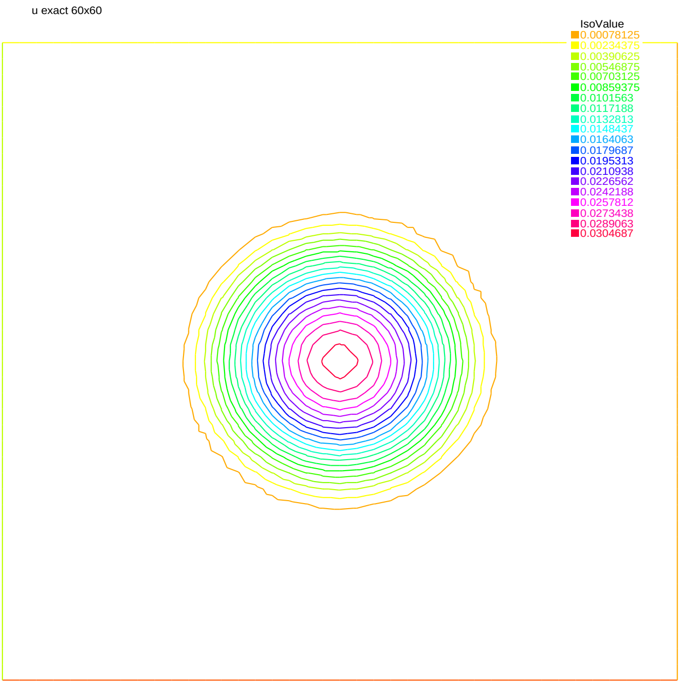


Fig. 4. Isovalues of exact u for $h = 1/30$.

classically known. Finally we have confirmed the theoretical results that we have established by numerical tests.

Finally we would like to confirm that the approach we have presented can be used successfully for other similar problems, in particular the bilateral obstacle problem.

Acknowledgement

The authors would like to thank the anonymous referees for their interesting remarks.

References

1. Kinderleher, D. Stampacchia. *An introduction to variational inequalities and their applications.* Academic Press (1980).
2. A. Addou, E. B. Mermri. *Sur une méthode de résolution d'un problème d'obstacle.* Math-Recherche & Application n°2, 59–69 (2000).
3. F. Brezzi & L. A. Caffarelli. *Convergence of the discrete free boundaries for*

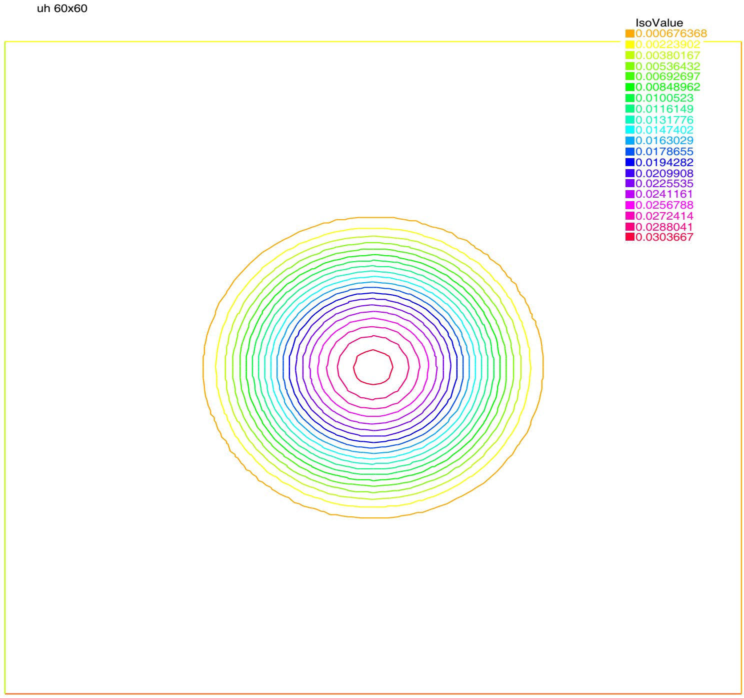


Fig. 5. Isovalues of u_h for $h = 1/30$.

finite element approximations. R.A.I.R.O Numerical Analysis vol. 17, n° 4, 385–395 (1983).

4. Ricardo H. Nochetto. *Sharp L^∞-Error Estimates for semilinear Elliptic Problems with free boundaries.* Numer. Maths. 54, 243-255 (1988).
5. R. Glowinski, J. L. Lions, R. Trémolières. *Analyse numérique des inéquations variationnelles* Tome*1.* Dunod, Paris (1976).
6. F. Brezzi, W. W. Hagar. *Error estimate for the finite element solution of variational inequality*, part II. Numer. Math. 31, 1–16 (1978).
7. Slimane, L; Bendali, A; Laborde, P. *Mixed formulations for a class of variational inequalities.* M2AN Math. Model. Numer. Anal. 38 (2004), no. 1, 177–201.
8. Degueil A. *Résolution par une méthode d'éléments finis d'un problème de STEPHAN en terme de température et en teneur en matériau non gelé.* Thèse 3$^{\text{ème}}$ cycle, Bordeaux (1977).
9. I. Ekland, R. Temam. *Analyse convexe et problèmes variationnels.* Dunod, Paris (1974).
10. Slimane, L. *Méthodes mixtes et traitement du verrouillage numérique pour la résolution des inéquations variationnelles.* Thèse de l'Institut National des Sciences Appliquées, TOULOUSE (2001).

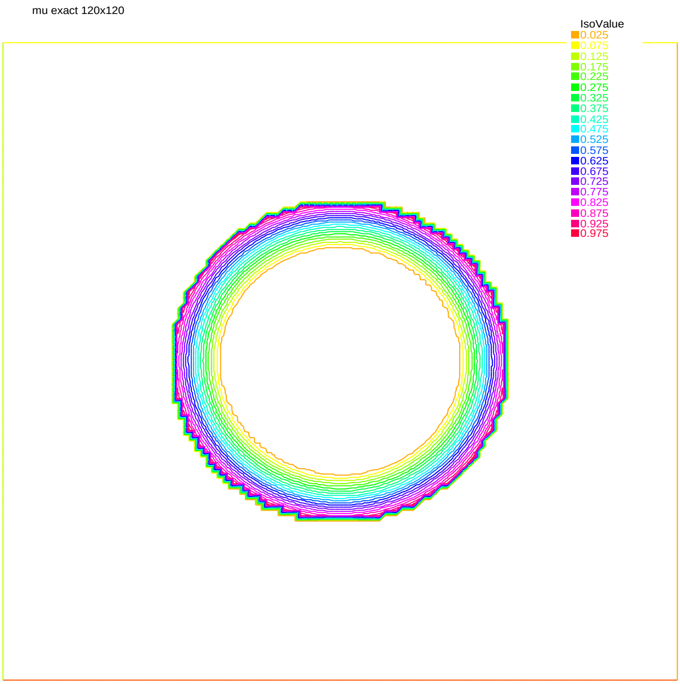


Fig. 6. Isovalues of exact μ for $h = 1/60$.

11. P. G. Ciarlet. *The Finite Element Method for Elliptic problems.* Amsterdam: North Holland, (1978).
12. P. A. Raviart, J. M. Thomas. *Introduction l'analyse numérique des équations aux dérivées partielles.* Masson, Paris (1992).
13. L. A. Caffarelli. *The Obstacle Problem revisted.* J. Fourier Anal. Appl. 4, 383–402 (1998).
14. F. Heicht, O. Pironneau, A. Le Hyaric, K. Ohtsuka. *Freefem++.* Laboratoire Jacques-Louis Lions, Université Pierre et Marie Curie, Paris, (http://www.freefem.org/ff++).

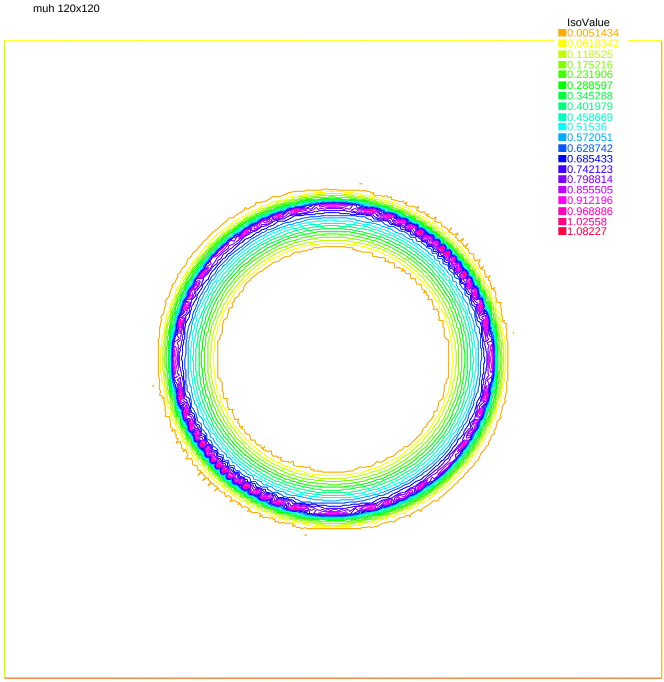


Fig. 7. Isovalues of μ_h for $h = 1/60$.

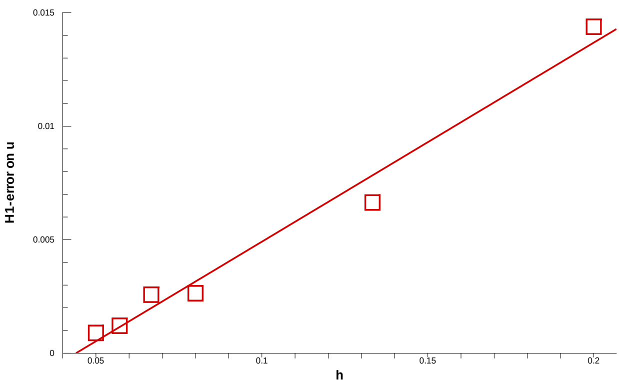


Fig. 8. The $H_0^1(\Omega)$-error on u for different values of h.

An obstacle problem via a sequence of penalized problems

E. Azroul and M. Rhoudaf

Département de Mathématiques et Informatique
Faculté des Sciences Dhar-Mahraz
B.P 1796 Atlas Fès, Morocco

In this paper, we shall concern with the existence result of unilateral parabolic degenerated problems associated to the equations of the form

$$\frac{\partial u}{\partial t} + A(u) = f \quad \text{in} \quad Q_T,$$

where A is a classical Leray-Lions operator acting from the weighted Sobolev space $L^p(0,T,W_0^{1,p}(\Omega,w))$ into its dual $L^{p'}(0,T,W^{-1,p'}(\Omega,w^*))$, while the datum f is assumed in $L^1(Q_T)$.
The proof is based on the penalty methods.

Keywords: Unilateral parabolic degenerate problem; Existence result; Penalty methods.

1. Introduction

In this paper, we investigate the problem of existence of solutions of the obstacle problems associated to the following initial-boundary value problem:

$$(P_e)\begin{cases} \frac{\partial u}{\partial t} - \operatorname{div}(a(x,t,u,\nabla u)) = f & \text{in} \quad Q_T = \Omega \times (0,T) \\ u = 0 & \text{on} \quad \Sigma = \partial\Omega \times (0,T) \\ u(0) = u_0 & \text{in} \quad \Omega, \end{cases}$$

where Ω is an open bounded subset of $I\!R^N$, $N \geq 1$, $T > 0$, and we have set Q_T the cylinder $\Omega \times (0,T)$ and Σ its lateral surface.

We assume that $a(x,t,\xi) : Q_T \times I\!R^N \to I\!R^N$ is a Carathéodry function (i.e., measurable with respect to (x,t) and continuous with respect to ξ) satisfying the hypotheses (H_2) below.

The data are taken such that: $f \in L^1(Q_T)$, $u_0 \in L^1(\Omega)$ and $u_0 \geq 0$. More precisely, this paper deals with the existence of solution to the obstacle degenerated parabolic problem (P_e) in the sense of entropy solution:

$$(P_u)\begin{cases} u \geq \psi \quad \text{a.e. in} \quad Q_T \\ T_k(u) \in L^p(0,T,W_0^{1,p}(\Omega,w)), \ u \in C([0,T],L^1(\Omega)) \\ \displaystyle\int_\Omega S_k(u-\varphi)(\tau)\,dx + \int_{Q_\tau} \frac{\partial \varphi}{\partial t} T_k(u-\varphi)\,dx\,dt \\ \qquad \leq \displaystyle\int_{Q_\tau} f T_k(u-\varphi)\,dx\,dt + \int_\Omega S_k(u_0-\varphi(x,0))\,dx \\ \varphi \in K_\psi \cap L^\infty(Q_T) \cap C([0,T],L^1(\Omega)) \quad \text{such that} \\ \qquad \frac{\partial \varphi}{\partial t} \in L^{p'}(0,T,W^{-1,p'}(\Omega,w^*)), \quad \forall\, k>0 \end{cases}$$

where $S_k(t) = \displaystyle\int_0^t T_k(s)\ ds$, $\psi \in L^\infty(\Omega) \cap W_0^{1,p}(\Omega,w)$ and $K_\psi = \{u \in L^p(0,T,W_0^{1,p}(\Omega,w)),\ u \geq \psi \ \text{ a.e. in } \ Q_T\}$.

The aim of our work is to investigate the relationship between the possibility to find solutions of (P) by approximating the singular data f and u_0 with sequences of regular functions. More precisely letting $\{f_\epsilon\}$ and u_0^ϵ be a standard approximation of f and u_0 (that is $f_\epsilon \to f$ in $L^1(Q)$ and $u_0^\epsilon \to u_0$ in $L^1(\Omega)$), and considering the approximate problem:

$$(P_e^\epsilon)\begin{cases} \frac{\partial u_\epsilon}{\partial t} - \operatorname{div}(a(x,t,u_\epsilon,\nabla u_\epsilon)) - \frac{1}{\epsilon}T_\epsilon(u_\epsilon-\psi)^- = f_\epsilon & \text{in} \quad Q_T \\ u_\epsilon = 0 & \text{on} \quad \Sigma \\ u_\epsilon(0) = u_0^\epsilon & \text{in} \quad \Omega. \end{cases}$$

We study the possibility to find a solution of (P_u) as a limit of a subsequence $\{u_\epsilon\}$ of solutions of (P_e^ϵ).

The penalized term $\frac{1}{\epsilon}T_\epsilon(u_\epsilon-\psi)^-$ introduced in (P_e^ϵ) play a crucial role in the proof of our main result, in particular this term allows to prove that the solution u of (P_u) belongs in K_ψ (that is $u \geq \psi$).

A priori estimates of the truncations $T_k(u_\epsilon)$ are obtained in $L^p(0,T,W_0^{1,p}(\Omega,w))$. For the passage to the limit, we prove the strong converge of the truncation of u_ϵ and the almost everywhere convergence of ∇u_ϵ is proved. An example of operator model is,

$$A(u) = -\operatorname{div}(|x|^r|\nabla u|^{p-2}\nabla u), \quad r>0.$$

In this context of parabolic problems, if $w \equiv 1$, existence results for (P_e) have been proved in[18] when f belongs to $L^p(0,T,W^{-1,p'}(\Omega))$ and u_0 is in $L^2(\Omega)$. The case where f belongs to $L^1(Q_T)$ is investigated in [,[1920]] while the case $w \neq 1$, is studied in [,[222]] where in the first work the authors have studied the variational case ($f \in L^{p'}(0,T,W^{-1,p'}(\Omega,w^*))$ and in the second work the $L^1(Q_T)$ case is treated.

Let us mention that in the literature of unilateral problems, the elliptic case is more studied, while the study of parabolic case is poor.

2. Preliminaries and basic assumptions

Let Ω be a bounded open subset of $I\!R^N$, p be a real number such that $1 < p < \infty$ and $w = \{w_i(x),\ 1 \le i \le N\}$ be a vector of weight functions, i.e., every component $w_i(x)$ is a measurable function which is strictly positive a.e. in Ω. Further, we suppose in all our considerations that, there exists

$$r_0 > \max(N,p) \text{ such that } \ w_i^{\frac{r_0}{r_0-p}} \in L^1_{loc}(\Omega), \tag{1}$$

and

$$w_i^{\frac{-1}{p-1}} \in L^1_{loc}(\Omega), \tag{2}$$

for any $0 \le i \le N$.

We denote by $W^{1,p}(\Omega,w)$ the space of all real-valued functions $u \in L^p(\Omega,w_0)$ such that the derivatives in the sense of distributions fulfill

$$\frac{\partial u}{\partial x_i} \in L^p(\Omega,w_i) \ \text{ for all } \ i = 1,...,N.$$

Which is a Banach space under the norm,

$$\|u\|_{1,p,w} = \left[\int_\Omega |u(x)|^p w_0 \, dx + \sum_{i=1}^N \int_\Omega |\frac{\partial u(x)}{\partial x_i}|^p w_i(x) \, dx\right]^{\frac{1}{p}}. \tag{3}$$

The condition (1) implies that $C_0^\infty(\Omega)$ is a subset of $W^{1,p}(\Omega,w)$ and consequently, we can introduce the subspace $W_0^{1,p}(\Omega,w)$ of $W^{1,p}(\Omega,w)$ as the closure of $C_0^\infty(\Omega)$ with respect to the norm (3). Moreover, the condition (2) implies that $W^{1,p}(\Omega,w)$ as well as $W_0^{1,p}(\Omega,w)$ are reflexive Banach spaces.

We recall that the dual space of weighted Sobolev spaces $W_0^{1,p}(\Omega,w)$ is equivalent to $W^{-1,p'}(\Omega,w^*)$, where $w^* = \{w_i^* = w_i^{1-p'},\ \ i = 0,...,N\}$ and where p' is the conjugate of p, i.e., $p' = \frac{p}{p-1}$. For more details, we refer the reader to.[12]

Now we state the following assumptions:

Assumption ($\mathbf{H_1}$) For $2 \le p < \infty$, we suppose that the expression

$$|||u||| = \left(\sum_{i=1}^N \int_\Omega |\frac{\partial u}{\partial x_i}|^p w_i(x) \, dx\right)^{\frac{1}{p}} \tag{4}$$

is a norm on $W_0^{1,p}(\Omega, w)$ which is equivalent to (3) and that there exists a weight function σ on Ω such that,

$$\sigma \in L^1(\Omega) \text{ and } \sigma^{-1} \in L^1(\Omega). \tag{5}$$

We assume also the Hardy inequality,

$$\left(\int_\Omega |u(x)|^p \sigma \, dx\right)^{\frac{1}{p}} \leq c \left(\sum_{i=1}^N \int_\Omega |\frac{\partial u}{\partial x_i}|^p w_i(x) \, dx\right)^{\frac{1}{p}}, \tag{6}$$

holds for every $u \in W_0^{1,p}(\Omega, w)$ with a constant $c > 0$ independent of u. Moreover, the imbedding

$$W_0^{1,p}(\Omega, w) \hookrightarrow L^p(\Omega, \sigma) \tag{7}$$

expressed by the inequality (6) is compact.

Note that $(W_0^{1,p}(\Omega, w), |||.|||)$ is a uniformly convex (and thus reflexive) Banach space.

Remark 2.1. Assume that $w_0(x) \equiv 1$ and there exists $\nu \in \left]\frac{N}{P}, +\infty\right[\cap \left[\frac{1}{P-1}, +\infty\right[$ such that

$$w_i^{\frac{N}{N-1}}, \; w_i^{-\nu} \in L^1(\Omega) \text{ for all } i = 1, ..., N. \tag{8}$$

Note that the assumptions (1) and (8) imply that,

$$|||u||| = \left(\sum_{i=1}^N \int_\Omega |\frac{\partial u}{\partial x_i}|^p w_i(x) \, dx\right)^{\frac{1}{p}} \tag{9}$$

is a norm defined on $W_0^{1,p}(\Omega, w)$ and its equivalent to (3) and that, the imbedding

$$W_0^{1,p}(\Omega, w) \hookrightarrow\hookrightarrow L^p(\Omega) \tag{10}$$

is compact [see,[12] pp 46].
Thus the hypotheses (H_1) is satisfied for $\sigma \equiv 1$.

Assumption $(\mathbf{H_2})$

$$a(x, t, s, \xi).\xi \geq \alpha \sum_{i=1}^N w_i |\xi_i|^p, \tag{11}$$

where $c_1(x, t)$ is a positive function in $L^{p'}(Q)$, and α, β are strictly positive constants.

We recall that, for $k > 1$ and s in $I\!R$, the truncation is defined as

$$T_k(s) = \begin{cases} s \text{ if } & |s| \leq k \\ k\frac{s}{|s|} \text{ if } & |s| > k. \end{cases}$$

3. Some technical lemmas

3.1. *Some functional properties of time-regularization of a function u*

In order to deal with time derivative, we introduce a time mollification of a function u belonging in some weighted Lebesgue space. Thus we define for all $\mu \geq 0$ and all $(x,t) \in Q_T$,

$$u_\mu = \mu \int_{-\infty}^{t} \tilde{u}(x,s)\exp(\mu(s-t))\, ds \quad \text{where} \quad \tilde{u}(x,s) = u(x,s)\chi_{(0,T)}(s).$$

Proposition 3.1.

1) If $u \in L^p(Q_T, w_i)$, then, u_μ is measurable in Q_T, $\frac{\partial u_\mu}{\partial t} = \mu(u - u_\mu)$ and

$$\left(\int_Q |u_\mu|^p w_i(x)\, dx\, dt\right)^{\frac{1}{p}} \leq \left(\int_Q |u|^p w_i(x)\, dx\, dt\right)^{\frac{1}{p}},$$

i.e.,

$$\|u_\mu\|_{L^p(Q_T,w_i)} \leq \|u\|_{L^p(Q_T,w_i)}.$$

2) If $u \in W_0^{1,p}(Q_T, w)$, then $u_\mu \to u$ in $W_0^{1,p}(Q_T, w)$ as $\mu \to +\infty$.
3) If $u_n \to u$ in $W_0^{1,p}(Q_T, w)$, then $(u_n)_\mu \to u_\mu$ in $W_0^{1,p}(Q_T, w)$.

3.2. *Some weighted imbedding and compactness results*

In this section, we establish some imbedding and compactness results in weighted Sobolev Spaces which allow in particular to extend in the settings of weighted Sobolev spaces, some trace results and the Aubin's and Simon's results[21].

Let $V = W_0^{1,p}(\Omega, w)$, $H = L^2(\Omega, \sigma)$ and let $V^* = W^{-1,p'}(\Omega, w^*)$, with $(2 \leq p < \infty)$.
Let $X = L^p(0,T,V)$. The dual space of X is $X^* = L^{p'}(0,T,V^*)$ where $\frac{1}{p'} + \frac{1}{p} = 1$ and denoting the space $W_p^1(0,T,V,H) = \{v \in X : v' \in X^*\}$ endowed with the norm

$$\|u\|_{w_p^1} = \|u\|_X + \|u'\|_{X^*}, \tag{12}$$

which is a Banach space. Here u' stands for the generalized derivative of u, i.e.,

$$\int_0^T u'(t)\varphi(t)\,dt = -\int_0^T u(t)\varphi'(t)\,dt \quad \text{for all} \quad \varphi \in C_0^\infty(0,T).$$

Lemma 3.1.

The Banach space H is an Hilbert space and its dual H' can be identified with him self, i.e., $H' \simeq H$.

Indeed, let

$$\begin{aligned} F : H \times H &\to I\!R \\ (f,g) &\mapsto \int_\Omega fg\sigma\,dx. \end{aligned}$$

Remark that F is a symmetric bilinear form, which is also continuous and defined positively, since

$$\int_\Omega fg\sigma\,dx = \int_\Omega f\sigma^{\frac{1}{2}}g\sigma^{\frac{1}{2}}\,dx \le \left(\int_\Omega |f|^2\sigma\,dx\right)^{\frac{1}{2}}\left(\int_\Omega |g|^2\sigma\,dx\right)^{\frac{1}{2}}.$$

Then, the Banach space H is an Hilbert space.
Finally by a standard argument, we can identified H with its dual H' i.e., $H' \simeq H$.

Lemma 3.2.[2]

The evolution triple $V \subseteq H \subseteq V^*$ is verified.

Lemma 3.3.[2]

Assume that,

$$\frac{\partial u_n}{\partial t} = h_n + k_n \quad in \quad D'(\Omega),$$

where h_n and k_n are bounded respectively in $L^{p'}(0,T,W^{1,p'}(\Omega,w^*)$ and in $L^1(Q_T)$.
If u_n is bounded in $L^p(0,T,W_0^{1,p}(\Omega,w))$, then $u_n \to u$ in $L^p_{loc}(Q_T,\sigma)$.

Lemma 3.4.[2]

Let $g \in L^r(Q_T,\gamma)$ and let $g_n \in L^r(Q_T,\gamma)$, with $\|g_n\|_{L^r(Q_T,\gamma)} \le c, 1 < r < \infty$. If $g_n(x) \to g(x)$ a.e in Q_T, then $g_n \rightharpoonup g$ in $L^r(Q_T,\gamma)$, where $\rightharpoonup$ denotes weak convergence and γ is a weight function on Q_T.

Lemma 3.5.[2]

Assume that (H_1) *and* (H_2) *are satisfied and let* (u_n) *be a sequence in* $L^p(0,T,W_0^{1,p}(\Omega,w))$ *such that* $u_n \rightharpoonup u$ *weakly in* $L^p(0,T,W_0^{1,p}(\Omega,w))$ *and*

$$\int_Q [a(x,t,u_n,\nabla u_n) - a(x,t,u_n,\nabla u)][\nabla u_n - \nabla u]\, dxdt \to 0. \tag{13}$$

Then, $u_n \to u$ *in* $L^p(0,T,W_0^{1,p}(\Omega,w))$.

Lemma 3.6.[23] *Let* $V \subseteq H \subseteq V^*$ *be an evolution triple. Then the imbedding*

$$W_p^1(0,T,V,H) \hookrightarrow C([0,T]),H)$$

is continuous .

3.3. *Main results*

Theorem 3.1. *Let* $u_0 \in L^1(\Omega)$ *such that* $u_0 \geq 0$. *Assume that* (H_1) *and* (H_2) *hold true. Then there exists at last one solution* $u \in C([0,T];L^1(\Omega))$ *such that* $u(x,0) = u_0$ *a.e. and for all* $\tau \in]0,T]$,

$$\begin{cases} T_k(u) \in L^p(0,T,W_0^{1,p}(\Omega,w)), u \geq \psi \quad a.e.\ in \ \ \Omega \\ \displaystyle\int_\Omega S_k(u(\tau) - \varphi(\tau))\, dx + \langle \frac{\partial \varphi}{\partial t}, T_k(u-\varphi)\rangle_{Q_\tau} \\ \displaystyle+ \int_{Q_\tau} a(x,t,u,\nabla u)\nabla T_k(u-\varphi)\, dx\, dt \\ \displaystyle\qquad \leq \int_{Q_\tau} f T_k(u-\varphi)\, dx\, dt + \int_\Omega S_k(u_0 - \varphi(x,0))\, dx \end{cases}$$

$\forall\ k > 0$ *and* $\forall\ \varphi \in K_\psi \cap L^\infty(Q)$ *such that* $\frac{\partial \varphi}{\partial t} \in L^{p'}(0,T,W^{-1,p'}(\Omega,w^*))$, *where* $Q_\tau = \Omega \times]0,\tau[$.

Proof.

Step 1: A priori estimates

Consider the approximate problem

$$(P_\epsilon) \begin{cases} \frac{\partial u_\epsilon}{\partial t} - \operatorname{div}(a(x,t,u_\epsilon,\nabla u_\epsilon)) - \frac{1}{\epsilon} T_{\frac{1}{\epsilon}}(u_\epsilon - \psi)^- = f_\epsilon \\ u_\epsilon \in L^p(0,T,W_0^{1,p}(\Omega,w)),\ u_\epsilon(x,0) = u_0^\epsilon \end{cases}$$

where $f_\epsilon \to f$ strongly $L^1(Q)$, $u_0^\epsilon \to u_0$ strongly $L^1(\Omega)$.

Thanks to,[2] there exists at least one solution of the problem (P_ϵ).

By choosing $T_\gamma(u_\epsilon - T_\beta(u_\epsilon))$, $\beta \geq \|\psi\|_\infty$ as test function in (P_ϵ), we get

$$\langle \tfrac{\partial u_\epsilon}{\partial t}, T_\gamma(u_\epsilon - T_\beta(u_\epsilon))\rangle + \int_{\beta \leq |u_\epsilon| \leq \beta+\gamma} a(x,t,u_\epsilon,\nabla u_\epsilon)\nabla u_\epsilon \, dx \, dt$$
$$- \int_Q \frac{1}{\epsilon} T_{\frac{1}{\epsilon}}(u_\epsilon - \psi)^- T_\gamma(u_\epsilon - T_\beta(u_\epsilon)) \, dx \, dt = \int_Q f_\epsilon T_\gamma(u_\epsilon - T_\beta(u_\epsilon)) \, dx \, dt \tag{14}$$

On the one hand, we have

$$\langle \frac{\partial u_\epsilon}{\partial t}, T_\gamma(u_\epsilon - T_\beta(u_\epsilon))\rangle = \int_\Omega S_\gamma^\beta(u_\epsilon(T)) \, dx - \int_\Omega S_\gamma^\beta(u_\epsilon^0) \, dx \tag{15}$$

where $S_\gamma^\beta(s) = \int_0^s T_\gamma(t - T_\beta(t)) \quad dt$, and by using the fact that $\int_\Omega S_\gamma^\beta(u_\epsilon(T)) \, dx \geq 0$ and $| \int_\Omega S_\gamma^\beta(u_\epsilon^0) \, dx| \leq \gamma \|u_\epsilon^0\|$, we get

$$\alpha \int_{\beta \leq |u_\epsilon| \leq \beta+\gamma} \sum_{i=1}^N \left| \frac{\partial u_\epsilon}{\partial x_i} \right|^p w_i(x) \, dx \, dt$$
$$- \frac{1}{\epsilon} \int_Q T_{\frac{1}{\epsilon}}(u_\epsilon - \psi)^- T_\gamma(u_\epsilon - T_\beta(u_\epsilon)) \, dx \, dt \leq c\gamma, \quad \forall \ \epsilon > 0 \tag{16}$$

so that

$$- \int_Q \frac{1}{\epsilon} T_{\frac{1}{\epsilon}}(u_\epsilon - \psi)^- \frac{T_\gamma(u_\epsilon - T_\beta(u_\epsilon))}{\gamma} \, dx \, dt \leq c$$

since $- \int_Q \frac{1}{\epsilon} T_{\frac{1}{\epsilon}}(u_\epsilon - \psi)^- \frac{T_\gamma(u_\epsilon - T_\beta(u_\epsilon))}{\gamma} \, dx \, dt \geq 0$, for every $\beta \geq \|\psi\|_\infty$, we deduce by Fatou's lemma as $\gamma \to 0$ that

$$\int_Q \frac{1}{\epsilon} T_{\frac{1}{\epsilon}}(u_\epsilon - \psi)^- \leq c. \tag{17}$$

Using in (P_ϵ) the test function $T_\epsilon(u_\epsilon)_{\chi(0,\tau)}$, we get for every $\tau \in (0,T)$,

$$\int_\Omega S_k(u_\epsilon(\tau)) \, dx + \int_{Q_\tau} a(x,t,T_k(u_\epsilon),\nabla T_k(u_\epsilon))\nabla T_k(u_\epsilon) \, dx \, dt$$
$$- \tfrac{1}{\epsilon} \int_Q T_{\frac{1}{\epsilon}}((u_\epsilon - \psi)^-) T_k(u_\epsilon) \, dx \, dt \leq ck$$

which gives thanks to (17)

$$\int_\Omega S_k(u_\epsilon(\tau)) \, dx + \int_{Q_\tau} a(x,t,T_k(u_\epsilon),\nabla T_k(u_\epsilon)) \, dx \, dt \leq ck. \tag{18}$$

Then,

$$\alpha \int_Q \sum_{i=1}^N \left| \frac{\partial T_k(u_\epsilon)}{\partial x_i} \right|^p w_i(x) \, dx \, dt \leq ck, \quad \forall \ k \geq 1. \tag{19}$$

Hence, $T_k(u_\epsilon)$ is bounded in $L^p(0,T,W_0^{1,p}(\Omega,w))$.
Let $k>0$ large enough and B_R be a ball of Ω, we have,

$$\begin{aligned} & k \ \mathrm{meas}(\{|u_\epsilon|>k\}\cap B_R\times[0,T]) = \int_0^T\int_{\{|u_\epsilon|>k\}\cap B_R} |T_k(u_\epsilon)|\ dx\ dt \\ & \leq \int_0^T\int_{B_R} |T_k(u_\epsilon)|\ dx\ dt \\ & \leq \left(\int_Q |T_k(u_\epsilon)|^p w_0\ dx\ dt\right)^{\frac{1}{p}} \times \left(\int_0^T\int_{B_R} w_0^{1-p'}\ dx\ dt\right) \end{aligned} \tag{20}$$

then, thanks to (H_1), we deduce that,

$$\begin{aligned} k \ \mathrm{meas}(\{|u_\epsilon|>k\}\cap B_R\times[0,T]) & \leq c\left(\int_Q \sum_{i=1}^N \left|\frac{\partial T_k(u_\epsilon)}{\partial x_i}\right|^p w_i(x)\ dx\ dt\right)^{\frac{1}{p}} \\ & \leq ck^{\frac{1}{p}} \end{aligned} \tag{21}$$

which implies that,

$$\mathrm{meas}(\{|u_\epsilon|>k\}\cap B_R\times[0,T]) \leq \frac{c_1}{k^{1-\frac{1}{p}}} \quad , \qquad \forall\ k\geq 1.$$

So, we have,

$$\lim_{k\to+\infty}(\mathrm{meas}(\{(x,t)\in Q:\ |u_\epsilon|>k\}\cap B_R\times[0,T])=0 \tag{22}$$

uniformly with respect to ϵ. Consider now a function nondecreasing $\xi_k\in C^2(I\!R)$ such that

$$\begin{cases} \xi_k(s)=s & \text{for } \ |s|\leq\frac{k}{2} \\ \xi_k(s)=k & \text{for } \ |s|\geq k. \end{cases}$$

Multiplying the approximate equation by $\xi_k'(u_\epsilon)$, we get

$$\begin{gathered} \frac{\partial}{\partial t}(\xi_k(u_\epsilon)) - \mathrm{div}(a(x,t,u_\epsilon,\nabla u_\epsilon)\xi_k'(u_\epsilon)) + a(x,t,u_\epsilon,\nabla u_\epsilon)\xi_k''(u_\epsilon) \\ -\frac{1}{\epsilon}T_{\frac{1}{\epsilon}}((u_\epsilon-\psi)^-)\xi_k'(u_\epsilon) = f_\epsilon\xi_k'(u_\epsilon), \end{gathered}$$

in the sense of distribution.
This implies, thanks to (19) and the fact that ξ_k' has compact support, that $\xi_k(u_\epsilon)$ is bounded in $L^p(0,T,W_0^{1,p}(\Omega,w))$, while it's time derivative $\frac{\partial}{\partial t}(\xi_k(u_\epsilon))$ is bounded in $L^{p'}(0,T,W^{-1,p'}(\Omega,w^*))+L^1(Q_T)$, hence lemma 3.3 allows us to conclude that $\xi_k(u_\epsilon)$ is compact in $L^p_{loc}(Q_T,\sigma)$.
Thus, for a subsequence, it also converges in measure and almost every where in Q_T since we have, for every $\lambda>0$

$$\begin{aligned}&\text{meas}(\{|u_\epsilon - u_\eta| > \lambda\} \cap B_R \times [0,T]) \leq \text{meas}(\{|u_\epsilon| > \tfrac{k}{2}\} \cap B_R \times [0,T])\\&+\text{meas}(\{|u_\eta| > \tfrac{k}{2}\} \cap B_R \times [0,T])\\&+\text{meas}(\{|\xi_k(u_\epsilon) - \xi_k(u_\eta)| > \lambda\} \cap B_R \times [0,T]).\end{aligned} \tag{23}$$

Let $\sigma > 0$, then, by (22) and the fact that $\xi_k(u_\epsilon)$ is compact in $L^p_{loc}(Q_T, \sigma)$, there exists $k(\sigma) > 0$ such that,

$$\text{meas}(\{|u_\epsilon - u_\eta| > \lambda\} \cap B_R \times [0,T]) \leq \sigma \quad \text{for all} \quad \epsilon, \eta \leq \epsilon_0(k(\sigma), \lambda, R).$$

This proves that (u_ϵ) is a Cauchy sequence in measure in $B_R \times [0,T]$, thus converges almost everywhere to some measurable function u. Then for a subsequence denoted again u_ϵ, we can deduce from (19) that,

$$T_k(u_\epsilon) \rightharpoonup T_k(u) \quad \text{weakly in} \quad L^p(0,T, W_0^{1,p}(\Omega, w)). \tag{24}$$

and then, the compact imbedding (7) gives,

$$T_k(u_\epsilon) \to T_k(u) \quad \text{strongly in} \quad L^p(Q_T, \sigma) \quad \text{and a.e. in} \quad Q_T. \tag{25}$$

Step 2: About the gradient of approximate solutions

In the sequel and throughout the paper, we will denote $\alpha(\epsilon, \mu, s)$ all quantities (possibly different) such that,

$$\lim_{s\to\infty} \lim_{\mu\to\infty} \lim_{\epsilon\to+0} \alpha(\epsilon, \mu, s) = 0.$$

Taking now $T_\eta(u_\epsilon - (T_k(u))_\mu)$, $\eta > 0$ as test function in (P_ϵ), we get

$$\begin{aligned}\langle \tfrac{\partial u_\epsilon}{\partial t}, T_\eta(u_\epsilon - (T_k(u))_\mu)\rangle + \int_Q a(x,t,u_\epsilon,\nabla u_\epsilon)\nabla T_\eta(u_\epsilon - (T_k(u))_\mu)\\ -\tfrac{1}{\epsilon}\int_{Q_T} T_{\frac{1}{\epsilon}}((u_\epsilon - \psi)^-) T_\eta(u_\epsilon - (T_k(u))_\mu)\, dx\, dt \leq c\eta,\end{aligned}$$

which implies that,

$$\begin{aligned}\langle \tfrac{\partial u_\epsilon}{\partial t}, T_\eta(u_\epsilon - (T_k(u))_\mu)\rangle + \int_{Q_T} a(x,t,u_\epsilon,\nabla u_\epsilon)\nabla T_\eta(u_\epsilon - (T_k(u))_\mu)\\ \leq \tfrac{\eta}{\epsilon}\int_{Q_T} T_{\frac{1}{\epsilon}}((u_\epsilon - \psi)^-)\, dx\, dt + c\eta\end{aligned}$$

and by (17)

$$\begin{aligned}\langle \tfrac{\partial u_\epsilon}{\partial t}, T_\eta(u_\epsilon - T_k(u)_\mu)\rangle + \int_{Q_T} a(x,t,u_\epsilon,\nabla u_\epsilon)\nabla T_\eta(u_\epsilon - (T_k(u)_\mu)\\ \leq c\eta.\end{aligned} \tag{26}$$

The first term of the left-hand side of the last inequality reads as,

$$\begin{aligned}\langle \tfrac{\partial u_\epsilon}{\partial t}, T_\eta(u_\epsilon - T_k(u)_\mu)\rangle = \langle \tfrac{\partial u_\epsilon}{\partial t} - \tfrac{\partial T_k(u)_\mu}{\partial t}, T_\eta(u_\epsilon - T_k(u)_\mu)\rangle\\ +\langle \tfrac{\partial T_k(u)_\mu}{\partial t}, T_\eta(u_\epsilon - T_k(u)_\mu)\rangle.\end{aligned} \tag{27}$$

The second term of the last equality can be written as,

$$\begin{aligned} &\langle \tfrac{\partial u_\epsilon}{\partial t} - \tfrac{\partial T_k(u)_\mu}{\partial t}, T_\eta(u_\epsilon - T_k(u)_\mu)\rangle = \int_\Omega S_\eta(u_\epsilon(T) - T_k(u)_\mu(T))\, dx \\ &- \int_\Omega S_\eta(u_0^\epsilon)\, dx \geq -\eta \int_\Omega |u_0^\epsilon|\, dx \\ &\geq -\eta c. \end{aligned} \tag{28}$$

The third term can be written as,

$$\langle \frac{\partial T_k(u)_\mu}{\partial t}, T_\eta(u_\epsilon - T_k(u)_\mu)\rangle = \mu \int_{Q_T} (T_k(u) - T_k(u)_\mu)(T_\eta(u_\epsilon - T_k(u)_\mu))\, dx\, dt \tag{29}$$

thus by letting $\epsilon \to 0$ and by using Lebesgue theorem,

$$\begin{aligned} &\int_{Q_T} (T_k(u) - T_k(u)_\mu)(T_\eta(u_\epsilon - T_k(u)_\mu))\, dx\, dt \\ &= \int_{Q_T} (T_k(u) - T_k(u)_\mu)(T_\eta(u - T_k(u)_\mu))\, dx\, dt. \end{aligned}$$

Consequently,

$$\langle \frac{\partial u_\epsilon}{\partial t}, T_\eta(u_\epsilon - T_k(u)_\mu)\rangle \geq \alpha(\epsilon, \mu) - \eta c \tag{30}$$

on the other hand,

$$\begin{aligned} &\int_{Q_T} a(x,t,u_\epsilon,\nabla u_\epsilon)\nabla T_\eta(u_\epsilon - T_k(u)_\mu)\, dx\, dt \\ &= \int_{\{|u_\epsilon - T_k(u)_\mu| < \eta\}} a(x,t,u_\epsilon,\nabla u_\epsilon)(\nabla u_\epsilon - \nabla T_k(u)_\mu)\, dx\, dt \\ &= \int_{\{|T_k(u_\epsilon) - T_k(u)_\mu| < \eta\}} a(x,t,T_k(u_\epsilon),\nabla T_k(u_\epsilon))(\nabla T_k(u_\epsilon) - \nabla T_k(u)_\mu)\, dx\, dt \\ &\quad + \int_{\{|u_\epsilon| > k\} \cap \{|u_\epsilon - T_k(u)_\mu| < \eta\}} a(x,t,u_\epsilon,\nabla u_\epsilon)(\nabla u_\epsilon - \nabla T_k(u)_\mu)\, dx\, dt \end{aligned}$$

which implies, by using the fact that

$$\int_{\{|u_\epsilon| > k\} \cap \{|u_\epsilon - T_k(u)_\mu| < \eta\}} a(x,t,u_\epsilon,\nabla u_\epsilon)\nabla u_\epsilon\, dx\, dt \geq 0,$$

that

$$\begin{aligned} &\int_{\{|T_k(u_\epsilon) - T_k(u)_\mu| < \eta\}} a(x,t,T_k(u_\epsilon),\nabla T_k(u_\epsilon))(\nabla T_k(u_\epsilon) - \nabla T_k(u)_\mu)\, dx\, dt \\ &\leq c\eta + \int_{\{|u_\epsilon| > k\} \cap \{|u_\epsilon - T_k(u)_\mu| < \eta\}} a(x,t,u_\epsilon,\nabla u_\epsilon)|\nabla T_k(u)_\mu|\, dx\, dt. \end{aligned} \tag{31}$$

Since $a(x,t,T_{k+\eta}(u_\epsilon),\nabla T_{k+\eta}(u_\epsilon))$ is bounded $\prod_{i=1}^{N} L^{p'}(Q_T,w_i^*)$, there exists some $h_{k+\eta}\in\prod_{i=1}^{N} L^{p'}(Q_T,w_i^*)$ such that,

$$a(x,t,T_{k+\eta}(u_\epsilon),\nabla T_{k+\eta}(u_\epsilon))\rightharpoonup h_{k+\eta} \quad \text{weakly in} \quad \prod_{i=1}^{N} L^{p'}(Q_T,w_i^*).$$

Consequently,

$$\begin{aligned}&\int_{\{|u_\epsilon|>k\}\cap\{|u_\epsilon-T_k(u)_\mu)|<\eta\}} a(x,t,u_\epsilon,\nabla u_\epsilon)|\nabla T_k(u)_\mu|\,dx\,dt\\ &\quad=\int_{\{|u|>k\}\cap\{|u-T_k(u)_\mu)|<\eta\}} h_{k+\eta}|\nabla T_k(u)_\mu|\,dx\,dt+\alpha(\epsilon)\end{aligned}$$

thanks to proposition 3.1, one easily has,

$$\int_{\{|u|>k\}\cap\{|u-T_k(u)_\mu)|<\eta\}} h_{k+\eta}|\nabla T_k(u)_\mu|\,dx\,dt+\alpha(\epsilon)=\alpha(\epsilon,\mu).$$

Hence,

$$\begin{aligned}&\int_{\{|T_k(u_\epsilon)-T_k(u)_\mu)|<\eta\}} a(x,t,T_k(u_\epsilon),\nabla T_k(u_\epsilon))(\nabla T_k(u_\epsilon)-\nabla T_k(u)_\mu)\,dx\,dt\\ &\qquad\le c\eta+\alpha(\epsilon,\mu).\end{aligned}\tag{32}$$

On the other hand, note that

$$\begin{aligned}&\int_{\{|T_k(u_\epsilon)-T_k(u)_\mu)|<\eta\}} a(x,t,T_k(u_\epsilon),\nabla T_k(u_\epsilon))(\nabla T_k(u_\epsilon)-\nabla T_k(u)_\mu)\,dx\,dt\\ &=\int_{\{|T_k(u_\epsilon)-T_k(u)_\mu)|<\eta\}} a(x,t,T_k(u_\epsilon),\nabla T_k(u_\epsilon))(\nabla T_k(u_\epsilon)-\nabla T_k(u))\,dx\,dt\\ &\quad+\int_{\{|T_k(u_\epsilon)-T_k(u)_\mu)|<\eta\}} a(x,t,T_k(u_\epsilon),\nabla T_k(u_\epsilon))(\nabla T_k(u)-\nabla T_k(u)_\mu)\,dx\,dt\end{aligned}\tag{33}$$

the last integral tends to 0 as $\epsilon\to 0$ and $\mu\to\infty$.
Indeed, we have that

$$\begin{aligned}&\int_{\{|T_k(u_\epsilon)-T_k(u)_\mu)|<\eta\}} a(x,t,T_k(u_\epsilon),\nabla T_k(u_\epsilon))(\nabla T_k(u)-\nabla T_k(u)_\mu)\,dx\,dt\to\\ &\int_{\{|T_k(u)-T_k(u)_\mu)|<\eta\}} h_k(\nabla T_k(u)-\nabla T_k(u)_\mu)\,dx\,dt \quad\text{as}\quad \epsilon\to 0.\end{aligned}$$

It is obviously that,

$$\int_{\{|T_k(u)-T_k(u)_\mu)|<\eta\}} h_k(\nabla T_k(u)-\nabla T_k(u)_\mu)\,dx\,dt\to 0 \quad\text{as}\quad \mu\to\infty.$$

We deduce then that,

$$\int_{\{|T_k(u_\epsilon)-T_k(u)_\mu)|<\eta\}} a(x,t,T_k(u_\epsilon),\nabla T_k(u_\epsilon))(\nabla T_k(u_\epsilon)-\nabla T_k(u))\,dx\,dt \\ \leq c\eta+\alpha(\epsilon,\mu). \tag{34}$$

Let A_ϵ be expression in brace above, then for any $0<\eta<1$

$$I_\epsilon=\int_{\{|T_k(u_\epsilon)-T_k(u)_\mu)|\leq\eta\}} A_\epsilon^\theta\,dx\,dt+\int_{\{|T_k(u_\epsilon)-T_k(u)_\mu)|>\eta\}} A_\epsilon^\theta\,dx\,dt$$

since, $a(x,T_k(u_\epsilon),\nabla T_k(u_\epsilon))$ is bounded in $\prod_{i=1}^N L^{p'}(Q_T,w_i^{1-p'})$, while $\nabla T_k(u_\epsilon)_{L_\infty}$ bounded in $\prod_{i=1}^N L^p(Q_T,w_i)$, then by applying the Hölder's inequality, we obtain,

$$I_\epsilon\leq c\left(\int_{\{|T_k(u_\epsilon)-T_k(u)_\mu)|<\eta\}} A_\epsilon\,dx\,dt\right)^\theta \\ +c_2\text{ meas}\{(x,t)\in Q_T:\ |T_k(u_\epsilon)-T_k(u)_\mu)|>\eta\}^{1-\theta}, \tag{35}$$

on the other hand, we have,

$$\begin{aligned}&\int_{\{|T_k(u_\epsilon)-T_k(u)_\mu)|<\eta\}} A_\epsilon\,dx\,dt\\ &=\int_{\{|T_k(u_\epsilon)-T_k(u)_\mu)|<\eta\}} a(x,t,T_k(u_\epsilon),\nabla T_k(u_\epsilon))(\nabla T_k(u_\epsilon)-\nabla T_k(u))\,dx\,dt\\ &-\int_{\{|T_k(u_\epsilon)-T_k(u)_\mu)|<\eta\}} a(x,t,T_k(u_\epsilon),\nabla T_k(u))(\nabla T_k(u_\epsilon)-\nabla T_k(u))\,dx\,dt\\ &=I_\epsilon^1+I_\epsilon^2\end{aligned} \tag{36}$$

using (34), we have,

$$I_\epsilon^1\leq c\eta+\alpha(\epsilon,\mu). \tag{37}$$

Concerning I_ϵ^2 the second term of the right hand side of the (36), it is easy to see that

$$I_\epsilon^2=\alpha(\epsilon) \tag{38}$$

because for all $i=1,\dots,N$, we have, $a_i(x,t,T_k(u_\epsilon),\nabla T_k(u))\to a_i(x,t,T_k(u),\nabla T_k(u))$ strongly in $L^{p'}(Q_T,w_i^{1-p'})$, while $\frac{\partial T_k(u_\epsilon)}{\partial x_i}\rightharpoonup\frac{\partial T_k(u)}{\partial x_i}$ weakly in $L^p(Q_T,w_i)$.

Combining (35), (36), (37) and (38) we get,

$$I_\epsilon\leq c\text{ meas}\{|T_k(u_\epsilon)-T_k(u)_\mu)|<\eta\}^\theta+c(\alpha(\epsilon,\mu,\eta))^{1-\theta}$$

and by passing to the limit sup over ϵ, μ and η

$$\lim_{\epsilon \to 0} \int_{Q_T} \Big\{ [a(x,t,T_k(u_\epsilon), \nabla T_k(u_\epsilon)) \\ -a(x,t,T_k(u_\epsilon), \nabla T_k(u))][\nabla T_k(u_\epsilon) - \nabla T_k(u)] \Big\}^\theta \, dx\, dt = 0 \tag{39}$$

and thus there exist a subsequence also denoted by u_ϵ such that,

$$\nabla u_\epsilon \longrightarrow \nabla u \text{ a.e. in } Q_T. \tag{40}$$

Step 3: Passage to the limit

Let $\varphi \in K_\psi \cap L^\infty(\bar{Q})$, choosing $T_k(u_\epsilon - \varphi)_{\chi(0,\tau)}$ as test function in (P_ϵ), we get,

$$\langle \tfrac{\partial u_\epsilon}{\partial t}, T_k(u_\epsilon - \varphi)\rangle_{Q_\tau} + \int_{Q_\tau} a(x,t,u_\epsilon,\nabla u_\epsilon)\nabla T_k(u_\epsilon - \varphi)\, dx\, dt \\ -\tfrac{1}{\epsilon}\int_{Q_\tau} T_{\frac{1}{\epsilon}}(u_\epsilon - \psi)^- T_\epsilon(u_\epsilon - \varphi)\, dx\, dt \\ = \int_{Q_\tau} f_\epsilon T_\epsilon(u_\epsilon - \varphi)\, dx\, dt \tag{41}$$

since, $-\frac{1}{\epsilon}\int_{Q_\tau} T_{\frac{1}{\epsilon}}(u_\epsilon - \psi)^- T_\epsilon(u_\epsilon - \varphi)\, dx\, dt \geq 0$ and $\frac{\partial u_\epsilon}{\partial t} = \frac{\partial}{\partial t}(u_\epsilon - \varphi) + \frac{\partial \varphi}{\partial t}$, we get

$$\int_\Omega S_k(u_\epsilon(\tau) - \varphi(\tau))\, dx + \langle \frac{\partial \varphi}{\partial t}, T_k(u_\epsilon - \varphi)\rangle_{Q_\tau} \\ + \int_{Q_\tau} a(x,t,u_\epsilon,\nabla u_\epsilon)\nabla T_k(u_\epsilon - \varphi)\, dx\, dt \\ \leq \int_{Q_\tau} f_\epsilon T_\epsilon(u_\epsilon - \varphi)\, dx\, dt + \int_\Omega S_k(u_\epsilon(0) - \varphi(0))\, dx. \tag{42}$$

Lemma 3.7. *The sequence (u_ϵ) is a Cauchy sequence in $C([0,T], L^1(\Omega))$, moreover, $u \in C([0,T], L^1(\Omega))$ and (u_ϵ) converges to u in $C([0,T], L^1(\Omega))$.*

This lemma will be proved below.

Because of $u_\epsilon \to u$ in $C([0,T], L^1(\Omega))$, then $\forall\, \tau \leq T$, $u_\epsilon(t) \to u(t)$ in $L^1(\Omega)$, thus

$$\int_\Omega S_k(u_\epsilon(\tau) - \varphi(\tau))\, dx \to \int_\Omega S_k(u - \varphi)\, dx \tag{43}$$

and

$$\int_\Omega S_k(u_\epsilon(0) - \varphi(0))\, dx \to \int_\Omega S_k(u_0 - \varphi(0))\, dx.$$

Let $M = k + \|\varphi\|_\infty$, then, we can write,

$$\int_{Q_T} a(x,t,u_n,\nabla u_n)\nabla T_k(u_n-\varphi)\,dx\,dt$$
$$= \int_{Q_T} a(x,t,T_M(u_n),\nabla T_M(u_n))\nabla T_k(u_n-\varphi)\,dx\,dt,$$

which implies by Fatou's lemma (and the fact

$$a(x,t,T_M(u_n),\nabla T_M(u_n)) \rightharpoonup a(x,t,T_M(u),\nabla T_M(u))$$

weakly in $\prod_{i=1}^{N} L^{p'}(Q, w_i^{1-p'})$, that we can deduce,

$$\begin{aligned}\int_{Q_T} & a(x,t,T_M(u),\nabla T_M(u))\nabla T_k(u-\varphi) \\ &\le \liminf_{\epsilon\to 0}\int_\Omega a(x,t,T_M(u_\epsilon),\nabla T_M(u_\epsilon))\nabla T_k(u_\epsilon-\varphi)\,dx\,dt.\end{aligned} \tag{44}$$

Moreover, since $\frac{\partial\varphi}{\partial t} \in L^{p'}(0,T,W^{-1,p'}(\Omega,w^*))$ and $\nabla T_k(u_\epsilon-\varphi) \rightharpoonup \nabla T_k(u-\varphi)$ weakly in $\prod_{i=1}^{N} L^p(Q_T,w_i)$, we get,

$$\int_{Q_\tau} \frac{\partial\varphi}{\partial t} T_k(u_\epsilon-\varphi)\,dx\,dt \to \int_{Q_\tau} \frac{\partial\varphi}{\partial t} T_k(u-\varphi)\,dx\,dt \tag{45}$$

$$\int_Q f_\epsilon T_k(u_\epsilon-\varphi)\,dx\,dt \to \int_Q f T_k(u-\varphi)\,dx\,dt. \tag{46}$$

Finally, by (42)-(46) we get,

$$\begin{aligned}\int_\Omega & S_k(u_\epsilon(\tau)-\varphi(\tau))\,dx + \langle\frac{\partial\varphi}{\partial t}, T_k(u-\varphi)\rangle_{Q_\tau} \\ &+ \int_{Q_\tau} a(x,t,u,\nabla u)\nabla T_k(u-\varphi)\,dx\,dt \\ &\qquad \le \int_{Q_\tau} f T_k(u-\varphi)\,dx\,dt + \int_\Omega S_k(u(0)-\varphi(0))\,dx,\end{aligned}$$

which completes the proof of the theorem.

Proof of Lemma 3.7:

Since $T_l(u) \in K_\psi$, for every $l \ge \|\psi\|_\infty$.

Let $v_\mu^{i,l} = (T_l(u))_\mu + e^{-\mu t} T_l(\eta_i)$ with $\eta_i \ge 0$ converges to u_0 in $L^1(\Omega)$, as test function in (46),

$$\langle \tfrac{\partial u_\epsilon}{\partial t}, T_k(u_\epsilon - v_\mu^{i,l})\rangle_{Q_\tau} + \int_{Q_\tau} a(x,t,u_\epsilon,\nabla u_\epsilon)\nabla T_k(u_\epsilon - v_\mu^{i,l})\ dx\ dt$$
$$-\tfrac{1}{\epsilon}\int_{Q_\tau} T_{\frac{1}{\epsilon}}(u_\epsilon - \psi)^- T_k(u_\epsilon - v_\mu^{i,l})\ dx\ dt \tag{47}$$
$$= \int_{Q_\tau} f_\epsilon T_k(u_\epsilon - v_\mu^{i,l})\ dx\ dt$$

since,

$$\begin{aligned}\frac{\partial u_\epsilon}{\partial t} &= \frac{\partial}{\partial t}(u_\epsilon - v_\mu^{i,l}) + \frac{\partial}{\partial t}(v_\mu^{i,l})\\ &= \frac{\partial}{\partial t}(u_\epsilon - v_\mu^{i,l}) + \mu(T_l(u) - v_\mu^{i,l})\end{aligned}$$

we deduce,

$$\langle \frac{\partial u_\epsilon}{\partial t}, T_k(u_\epsilon - v_\mu^{i,l})\rangle_{Q_\tau} = \langle \frac{\partial}{\partial t}(u_\epsilon - v_\mu^{i,l}), T_k(u_\epsilon - v_\mu^{i,l})\rangle + \alpha(\epsilon,\mu,l)$$

which implies that,

$$\langle \frac{\partial}{\partial t}(u_\epsilon - v_\mu^{i,l}), T_k(u_\epsilon - v_\mu^{i,l})\rangle = \langle \frac{\partial u_\epsilon}{\partial t}, T_k(u_\epsilon - v_\mu^{i,l})\rangle_{Q_\tau} + \alpha(\epsilon,\mu,l) \tag{48}$$

on the other hand, by using the monotonicity of a and the fact that, $-\int_{Q_\tau}\frac{1}{\epsilon}T_{\frac{1}{\epsilon}}(u_\epsilon - \psi)^- T_k(u_\epsilon - v_\mu^{i,l})\ dx\ dt \geq 0$, we deduce that by (47),

$$\begin{aligned}\langle \tfrac{\partial u_\epsilon}{\partial t}, T_k(u_\epsilon - v_\mu^{i,l})\rangle_{Q_\tau} &+ \int_{Q_\tau} a(x,t,u_\epsilon,\nabla v_\mu^{i,l})\nabla T_k(u_\epsilon - v_\mu^{i,l})\ dx\ dt\\ &\leq \int_{Q_\tau} f_\epsilon T_k(u_\epsilon - v_\mu^{i,l})\ dx\ dt\end{aligned}$$

since for all $i = 1,...,N$ $a_i(x,t,T_{2k+l}(u_\epsilon),\nabla v_\mu^{i,l}) \to a_i(x,t,T_{k+2l}(u),\nabla v_\mu^{i,l})$ strongly in $L^{p'}(Q,w_i^{1-p'})$ while $\frac{\partial}{\partial x_i}T_k(u_\epsilon - v_\mu^{i,l}) \rightharpoonup \frac{\partial}{\partial x_i}T_k(u - v_\mu^{i,l})$ weakly in $L^p(Q,w_i)$, we have,

$$\langle \frac{\partial u_\epsilon}{\partial t}, T_k(u_\epsilon - v_\mu^{i,l})\rangle \leq \alpha(\epsilon,\mu,i,l). \tag{49}$$

Therefore, by writing

$$\begin{aligned}\int_\Omega S_k(u_\epsilon(\tau) - v_\mu^{i,l}(\tau))\ dx = \langle \frac{\partial u_\epsilon}{\partial t}, T_k(u_\epsilon - v_\mu^{i,l})\rangle_{Q_\tau} &- \langle \frac{\partial v_\mu^{i,l}}{\partial t}, T_k(u_\epsilon - v_\mu^{i,l})\rangle_{Q_\tau}\\ &+ \int_\Omega S_k(u_0 - T_l(\eta_i))\ dx\end{aligned} \tag{50}$$

and using (48) and (49) we get,

$$\int_\Omega S_k(u_\epsilon(\tau) - v_\mu^{i,l}(\tau))\,dx \le \alpha(\epsilon,\mu,i,l) \tag{51}$$

which implies, by writing,

$$2\int_\Omega S_k(\frac{u_\epsilon - u_\lambda}{2})\,dx \le \int_\Omega \Big(S_k(u_\epsilon(\tau) - v_\mu^{i,l}(\tau)) + S_k(u_\lambda(\tau) - v_\mu^{i,l}(\tau))\Big)\,dx,$$

that

$$\int_\Omega S_k(\frac{u_\epsilon - u_\lambda}{2})\,dx \le \alpha(\epsilon,\lambda). \tag{52}$$

Finally, by Hölder's inequality, we have,

$$\int_\Omega |u_\epsilon - u_\lambda|\,dx = \int_{\{|u_\epsilon - u_\lambda|\le 1\}} |u_\epsilon - u_\lambda|\,dx + \int_{\{|u_\epsilon - u_\lambda|>1\}} |u_\epsilon - u_\lambda|\,dx$$

$$\le \left(\int_{\{|u_\epsilon - u_\lambda|\le 1\}} |u_\epsilon - u_\lambda|^2\,dx\right)^{\frac{1}{2}} \mathrm{meas}(\Omega)^{\frac{1}{2}} + \int_{\{|u_\epsilon - u_\lambda|>1\}} |u_\epsilon - u_\lambda|\,dx$$

$$\le \mathrm{meas}(\Omega)^{\frac{1}{2}} \left(\int_{\{|u_\epsilon - u_\lambda|\le 1\}} 2S_1|u_\epsilon - u_\lambda|^2\,dx\right)^{\frac{1}{2}} + \int_{\{|u_\epsilon - u_\lambda|>1\}} 2S_1(u_\epsilon - u_\lambda)\,dx$$

since

$$(\frac{|y|}{2})\chi_{\{|y|>1\}} \le (\frac{|y|}{2} + \frac{|y|-1}{2})\chi_{\{|y|>1\}} = S_1(y)\chi_{\{|y|>1\}}$$

and $(\frac{|y|^2}{2})\chi_{\{|y|\le 1\}} = S_1(y)\chi_{\{|y|\le 1\}}$ then by (52) we deduce that, $\int_\Omega |u_\epsilon(\tau) - u_\lambda(\tau)|\,dx \le \alpha(\epsilon,\lambda)$, not depending on τ. And thus (u_ϵ) is a Cauchy sequence in $C([0,T], L^1(\Omega))$, and since $u_\epsilon \to u$, a.e. in Q, we deduce that $u_\epsilon \to u$, a.e. in $C([0,T], L^1(\Omega))$.

References

1. R. Adams, *Sobolev spaces*, AC, Press, New York, (1975)
2. L. Aharouch, E. Azroul and M. Rhoudaf *Strongly nonlinear variational parabolic problems in Weighted Sobolev spaces* to appear in AJMAA.
3. Y. Akdim, E. Azroul and A. Benkirane, *Existence Results for Quasilinear Degenerated Equations Via Strong Convergence of Truncations*, Revista Matematica Complutense 17, , N.2, (2004) pp 359-379.
4. Y. Akdim, E. Azroul and A. Benkirane, *Existence of Solution for quasilinear degenerated Elliptic Unilateral Problems*, Ann. Math. Blaise pascal vol 10 (2003) pp 1-20.
5. Y. Akdim, E. Azroul and A. Benkirane, *Existence of solution for quasilinear degenerated elliptic equation*, Electronic J. Diff. Equ. Vol 2001 , N71, (2001) pp 1-19.

6. H. BREZIS, and F.E. BROWDER, *Strongly nonlinear parabolic initial-boundary value problems* , Proc. Nat Acad. Sci. U. S. A. 76 (1976). pp. 38-40.
7. J. BERKOVITS, V. MUSTONEN, *Topological degree for perturbation of linear maximal monotone mappings and applications to a class of parabolic problems*,Rend.Mat.Roma,Ser,VII, 12 (1992), pp. 597-621.
8. L. BOCCARDO, F. MURAT, *Strongly nonlinear Cauchy problems with gradient dependt lower order nonlinearity*, Pitman Research Notes in Mathematics, 208 (1988), pp. 347-364.
9. L. BOCCARDO, F. MURAT, *Almost everywhere convergence of the gradients of solutions to elliptic and parabolic equations,* Nonlinear analysis, T.M.A., 19 (1992), n 6, pp. 581-597.
10. A. DALLAGLIO A. ORSINA , *Non linear parabolic equations with natural growth condition and L^1 data.* Nolinear Anal., T.M.A., 27 n1 (1996). pp. 59-73.
11. P. DRABEK, A. KUFNER and L. MUSTONEN, *Pseudo-monotonicity and degenerated or singular elliptic operators*, Bull. Austral. Math. Soc. Vol. 58 (1998), 213-221.
12. P. DRABEK, A. KUFNER and F. NICOLOSI, *Non linear elliptic equations, singular and degenerated cases*, University of West Bohemia, (1996).
13. A. KUFNER, Weighted Sobolev Spaces, John Wiley and Sons, (1985).
14. R. LANDES, *On the existence of weak solutions for quasilinear parabolic initial-boundary value problems,* Proc. Roy. Soc. Edinburgh sect. A. 89 (1981), 217-137.
15. R. LANDES, V. MUSTONEN, *A strongly nonlinear parabolic initial-boundary value problems*, Ark. f. Math. 25. (1987).
16. R. LANDES, V. MUSTONEN, *On parabolic initial-boundary value problems with critical growth for the gradient*, Ann. Inst. H. Poincaré11 (2) (1994) 135-158.
17. J. LERAY, J.L. LIONS, *Quelques resultats de Višik sur les problèmes elliptiques nonlinéaires par les méthodes de Minty-Browder*, Bull. Soc. Math. France 93 (1995), 97-107.
18. J.L. LIONS, *quelques methodes de résolution des problèmes aux limites non linéaires*, Dunod et Gauthiers-Villars, 1969.
19. A. PORRETTA *Existence results for nonlinear parabilc equations via strong convergence of truncations*, Ann. Mat. Pura. Appl. (1999), pp. 143-172.
20. J. M. RAKOTOSON *A Compactness lemma for quasilinear problems: application to parabolic equations* J. Funct. Anal. 106 (1992), pp. 358-374.
21. J. SIMON *Compact sets in the space $L^p(0,T,B)$*, Ann. Mat. Pura. Appl. 146 (1987), pp. 65-96.
22. M. Rhoudaf *Existence results for Strongly nonlinear degenerated parabolic equations via strong convergence of truncations with L^1-data* thèse de Doctorat de M.Rhoudaf, Univ. Sidi Mohamed Ben Abdella. Fès, Maroc 2006.
23. E. ZEIDLER, *nonlinear functional analysis and its applications, II A and II B, Springer-Verlag (New York-Heidlberg, 1990).*

Numerical analysis of slopes stability under seismic loading in Lebanon

Fadi Hage Chehade

Université Libanaise, Institut Universitaire de Technologie
BP 813 Saida, Lebanon
E-mail: fchehade@ul.edu.lb

Marwan Sadek

Laboratoire de Mécanique de Lille, UMR 8107
Université des Sciences et Technologies de Lille, France
E-mail: marwan.sadek@polytech-lille.fr

Most common methods used in engineering practice to assess the seismic stability of slope consists of a pseudo static approach or traditional approaches based on the limit equilibrium or on the hypothesis of failure calculation. Such methods are widely used because of their simplicity and there is no need of sophisticated software for their application. However, they neglect important elements such as the soil deformability, the dynamic amplification and non linear soil behavior. The present study concerns the analysis of the seismic slopes stability using global dynamic approach. It will mainly focus on dynamic amplification in the slope under real earthquake records with frequency content close to the natural frequency of the slope. Analysis is conducted by numerical modeling using Flac3D finite difference program. Results are presented in the case of linear visco-elastic then elastoplastic behaviour for earthfill materials in order to elucidate the influence of plasticity.

Keywords: Seismic slope; Numerical modeling; Elastoplastic behavior.

1. Introduction

Recent devastating earthquakes in Pakistan, Turkey, Algeria and recently in China call to the mind the high risk exposure of Lebanon. This country is located over active fault (Harajli et al., 2002) and many geologists and experts shared the view that a major seismic event may occur in Lebanon in the future. Moreover, many earthquakes (more than 500 according to the National Council of Scientific Research in Lebanon), of low magnitudes between three and five, have been registered in Lebanon between February and October 2008. These events lead to great anxiety among Lebanese population because of high risk exposure in case of earthquake. Indeed there is no sufficient protection in most of civil constructions to seismic loading. It is well known that the slopes represent a weak area where the seismic consequences could be amplified (Davidovici, 1999). Large movement such as soil collapse, bloc rocks fall, rock or soil sliding could arise in steep slopes (angle more than 35) in case of earthquake [Keefer, 1984 ; Rodriguez et al., 1999]. Lebanon topography contains a lot of slopes and actually they become highly occupied because of the population migration from urban to surrounded suburb areas. Around the capital Beirut, it can be seen several slopes zones with high density of population.

The most common method used in engineering practice to assess the seismic stability of slope consists on a pseudo static approach where the earthquake effect on a potential soil mass is represented by means of equivalent static horizontal force equal to the soil mass multiplied by a seismic coefficient. This approach is quite simplistic since it attempts to represent complex dynamic behaviour in terms of static forces. Stability is expressed in terms of an overall factor of safety. The implicit assumption is that the soil is rigid-perfectly plastic behaving as an undeformable block.

Other traditional approaches are based on the limit equilibrium or on the hypothesis of failure calculation (for example Bishop (1955)). They are widely used in engineering practice because of their simplicity and there is no need of sophisticated software for their application. The main assumptions are based on the soil behavior (rigid bloc) and on the failure modes and their localization (circular, plane, . . .).

Progress in the area of geotechnical computation and numerical modeling offers interesting facilities for the analysis of the seismic induced response of soil and structure systems in considering complex issues such as the soil non linearity, the evolution of the pore pressure and real earthquake records. Detailed analysis techniques include equivalent linear (decoupled) solutions, and non linear finite element and finite difference coupled or decoupled formulations (Lin and Chao 1990, Abouseeda and Dakoulas 1998, Cascone and Rampello 2003, . . .).

Wood (1973) showed that where the principal energy of the input motions approaches the fundamental frequency of the unrestrained backfill, dynamic amplification becomes an important factor, which is not considered in engineering approaches that assess the earth fill dam stability. The present paper proposes a numerical study of the seismic behaviour of slopes. It will mainly focus on dynamic amplification in the slope under earthquake loading with frequency content close to the natural frequency of the slope. In the present study, analysis is conducted using a finite difference modeling. Results are presented in the cases of linear visco-elastic and elastoplastic behaviours for earthfill materials in order to elucidate the influence of plasticity. Indeed, since we deal with unconfined material, the seismic loading generally induces plasticity in the soil. This plasticity can influence the all over response of the slope, because of its influence on damping and on the dominant frequencies.

2. Dynamic analysis

2.1. *Problem under consideration and basic equations*

The selected example is a simplified representation of typical slope geometry (Figure 1). Mechanical properties of the soil used in the analyses are presented in Table 1. They correspond to the case of one layer of unconsolidated weak soil. Numerical analyses are conducted using the finite difference FLAC3D program based on a continuum finite difference discretization using the Langrangian approach (Flac3D, 2005).

In the Lagrangian formulation adopted in FLAC3D, a point in the medium is characterized by the vector components x_i, u_i, v_i and $\dfrac{dv_i}{dt}$, $i = 1, 3$ of position, displacement, velocity and acceleration, respectively.

The state of stress at a given point of the medium is characterized by the symmetric stress tensor σ_{ij} . The traction vector $[t]$ on a face with unit normal $[n]$ is given by Cauchy's formulae

$$t_i = \sigma_{ij}.n_j.$$

In an infinitesimal time dt, the medium experiences an infinitesimal strain determined by the translations $v_i dt$, and the corresponding components of the strain-rate tensor may be written as

$$\xi_{ij} = \frac{1}{2}(v_{i,j} + v_{j,i})$$

where partial derivatives are taken with respect to components of the current position vector $[x]$. Application of the continuum form of the momentum principle yields Cauchy's equations of motion :

$$\sigma_{ij,j} + \rho b_i = \frac{\rho dv_i}{dt}$$

where ρ is the mass per unit volume of the medium, $[b]$ is the body force per unit mass, and $\dfrac{d[v]}{dt}$ is the material derivative of the velocity. These laws govern, in the mathematical model, the motion of an elementary volume of the medium from the forces applied to it.

FLAC3D is based on a continuum finite difference discretization using the Langrangian approach. Every derivative in the set of governing equations is replaced directly by an algebraic expression written in terms of the field variables (e.g. stress or displacement) at discrete point in space. An important aspect of the model is the inclusion of the equations of motion; the calculation sequence first invokes the equations of motion to derive new velocities and displacements from stresses and forces. Then, strain rates are derived from velocities, and new stresses from strain rates. Every cycle around the loop correspond to one time step. The final solution is reached (using a damped solution) when the body is in equilibrium or in steady-state flow (plastic flow), and the out of balance force goes to zero.

Dynamic loading is applied at the base of the foundation layer as a velocity excitation. Free field boundaries were applied at the sides of the model in order to stop boundary effect.

Kuhlemeyer and Lysmer (1973) showed that for an accurate representation of the wave transmission through the soil model, the spatial element size, Δl, must be smaller than approximately one-tenth to one-eighth of the wavelength associated with the highest frequency component of the input wave i.e.,

$$\Delta l \leq \frac{\lambda}{10}. \tag{1}$$

Here, λ is the wave length associated with the highest frequency component that contains appreciable energy. The consequence is that reasonable analyses may be time and memory consuming. In such cases, it may be possible to adjust the input by recognizing that most of the power for the input history is contained in lower frequency components ($< 10Hz$).

Rayleigh damping of 5% is used in the analysis to compensate the energy dissipation through the medium (Paolucci, 2002; Lokmer et al., 2002). When plasticity is considered, damping occurs mainly through hysteretic looping, Rayleigh damping is fixed at 2% ; the. The maximum length of element is fixed to $1m$ in both vertical and horizontal directions (Figure 2).

2.2. *Elastic response*

In order to quantify the influence of frequency loading, numerical simulations has been firstly conducted under harmonic loading composed of 15

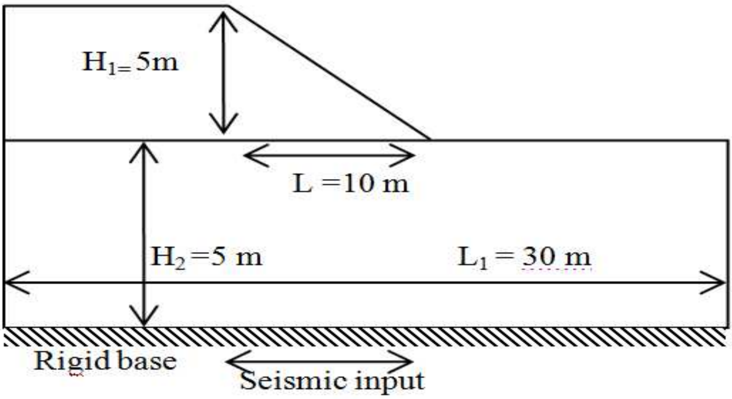


Fig. 1. A typical slope geometry.

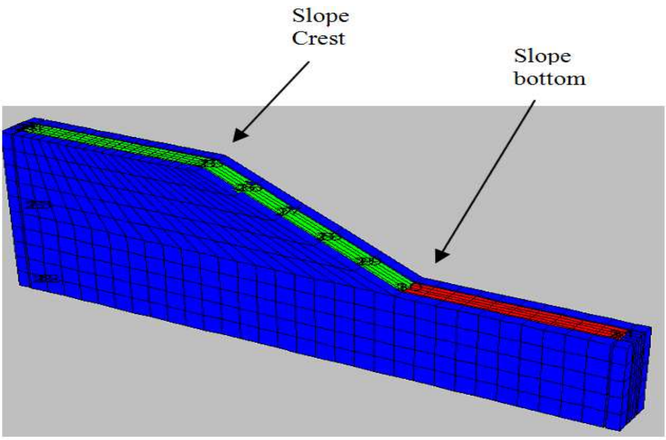


Fig. 2. Slope general configuration and finite difference mesh.

cycles. Figures 2 and 3 depict the maximum lateral amplifications according to the loading frequency applied at the base. It can be noted that where the applied frequency approach the fundamental frequency of the soil mass, the lateral amplification increase significantly from the bottom to the slope crest. For example, when the frequency loading fch equal the fundamental frequency of the system, the lateral amplification at the crest (10.3) is 3 times higher than that obtained at the bottom (3.4). In this

Young's Modulus (MPa)	Poisson's ratio	Unit weight (kN/m^3)	Plasticity		
			Cohesion (kPa)	Friction Angle (°)	Dilation Angle (°)
8	0.33	20	10	30	0

case, it's evident that neglecting soil deformability as used in pseudo-static approach, leads to significant error. From the other hand, when load frequency moves aside the fundamental frequency, lateral amplification shows little discrepancy between the top and the crest of the slope. In this case, the hypothesis of constant acceleration could be accepted as presumed in simplified approach.

2.3. *Influence of plasticity*

In order to examine the influence of plasticity, the slope system has been subjected to earthquake loading representative of the 1999 Kocaeli earthquake in Turkey ($Mw = 7.4$, Chen and Scawthorn 2003). Note that the dominant frequency of applied load is about 0.9 Hz which is close to the fundamental frequency of the system. Figure 4 shows a comparison between the elastic and elastoplastic analyses at the maximum lateral amplification. It can be observed that the plastic deformation leads to a decrease in the velocity amplification, in particular in the upper part (Figures 5 and 6). The reduction is dependent on the intensity of the applied loading (Figure 6). For example, it attains about 50% when the velocity (v) is equal to $2m/s$. This result could be attributed to the energy dissipation by plastic deformation and to the influence of plasticity on the reduction of the fundamental frequencies of the slope. When the applied velocity exceeds $0.5m/s$ which corresponds to a moderate earthquake, the lateral amplification show little discrepancy between the bottom and the top. For lower values, lateral amplification shows important difference with the depth and the hypothesis of constant soil amplification is not valid.

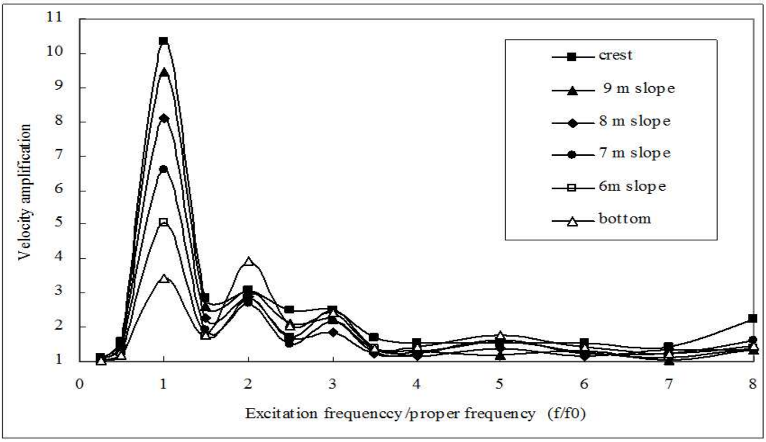


Fig. 3. Amplification at different positions of the slope according to the load frequency.

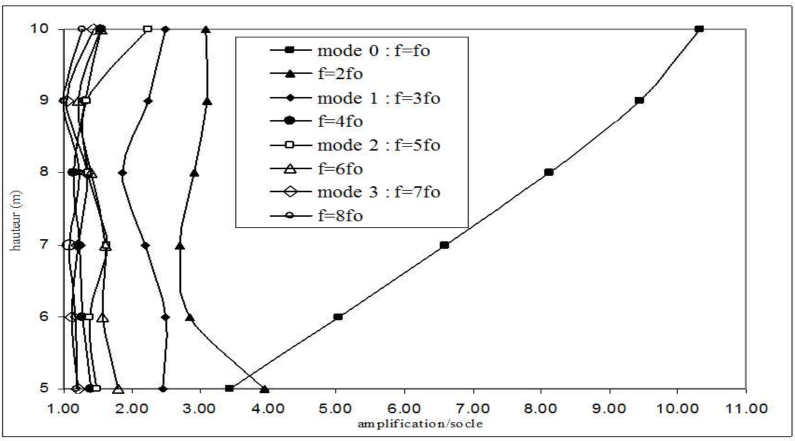


Fig. 4. Variation of maximum lateral amplification along the slope with frequency loading.

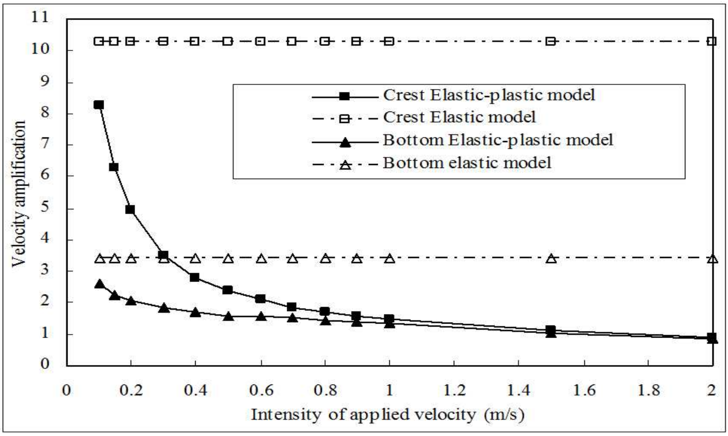


Fig. 5. A decrease in the velocity amplification.

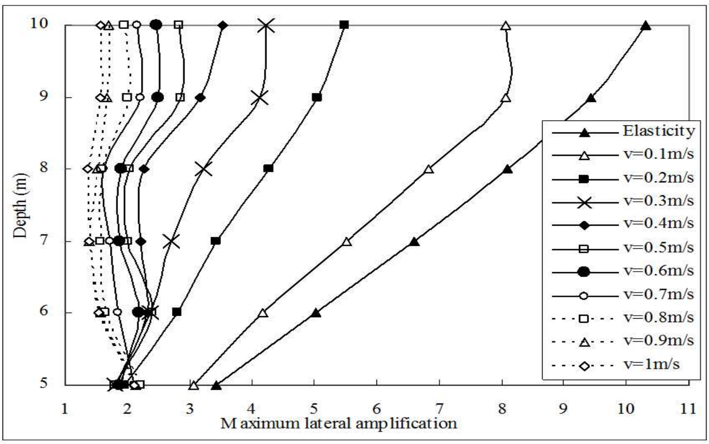


Fig. 6. Influence of plasticity on lateral amplification.

3. Conclusion

This paper included analysis of the seismic stability of slopes. Analyzes were conducted for harmonic and real earthquake records. For dynamic loading with dominant frequency close to the fundamental frequency of the slope, lateral amplification show a important discrepancy along the depth and the hypothesis of constant acceleration used in pseudo static approach is not valid. Elastoplastic analyzes show that the plastic deformation leads to a decrease in the velocity amplification, in particular in the upper part. Parametric analysis shows for moderate earthquake, the dynamic amplification still a important parameter and should be taken into account in any stability survey. Work is under progress to incorporate the variation of dynamic amplification in simplified approaches.

References

1. H. Abouseeda, and P. Dakoulas, Non-Linear Dynamic Earth Dam-Foundation Interaction Using a BE-FE Method, Earthquake Engineering and Structural Dynamics, Vol. 27 (1998), pp. 917-936.
2. A. Bishop, The use of the slip circle in the stability analysis of slopes, Geotechnique, 5(1) (1955), pp. 7-17.
3. E. Cascone, and S. Rampello, Decoupled Seismic Analysis of an Earth Dam, Soil Dynamics and Earthquake Engineering, Vol. 23 (2003), pp. 349-365.
4. W.F. Chen and C. Scawthorn, Earthquake Engineering Handbook, CRC Press LLC, 2003
5. V. Davidovici, la construction en zone sismique, Editions Le Moniteur, Paris (1999).
6. M. Harajli, S. Sadek and R. Aspahan, Evaluation of the seismic hazard of Lebanon, Journal of the Seismology, 6 (2002), pp. 257-277.
7. Itasca Consulting Group, FLAC: Fast Lagrangian Analysis of Continua, vol. I. User's Manual, vol. II. Verification Problems and Example Applications, Second Edition (FLAC3D Version 3.0), Minneapolis, Minnesota 55401 USA, 2005.
8. D.K. Keefer, Landslides caused by earthquakes, Bulletin of the seismological society of America, 95 (1984), pp. 406-421.
9. R. L.Kuhlmeyer and J. Lysmer,, Finite Element Method Accuracy for Wave Propagation Problems, J. Soil Mech. and Foundations Div., ASCE, 99(SM5) (1973), pp. 421-427.
10. J. Lin and B. Chao, Estimation of Shear Moduli and Damping Factors of Earth Dam Materials, Earthquake Engineering and Structural Dynamics, Vol. 19 (1990), pp. 891-910.
11. R. Paolucci, Amplification of earthquake ground motion by steep topographic irregularities", Earthquake Engineering and Structural dynamics, 2002; 31, pp.1831-1853.
12. C.E. Rodriguez, J. J. Bommer and R. J. Chandler, Earthquake-induced landslides: 1980-1997. Soil dynamics earthquake engineering, 18 (1999), pp. 325-346.
13. J. Wood, Earthquake-Induced Soil Pressures on Structures," Report EERL 73-05, California Institute of Technology, Pasadena 1973.

Existence and uniqueness result for a class of nonlinear parabolic equations with L^1 data

Hicham Redwane

Faculté des Sciences Juridiques, Economiques et Sociales
Université Hassan 1, B.P. 764, Settat, Morocco
E-mail: redwane_hicham@yahoo.fr

In this paper, we consider the initial-boundary value problem of a nonlinear parabolic equation with the data belong to L^1 and no growth assumption is made on the nonlinearities. We establish the existence and partial uniqueness theorems of renormalized solutions.

Keywords: Nonlinear parabolic equation; Renormalized solution; Existence and uniqueness.

1. Introduction

This paper is concerned with the following initial-boundary value problem

$$\frac{\partial b(x,u)}{\partial t} - div\Big(A(x,t)Du + \Phi(u)\Big) + f(x,t,u) = 0 \quad \text{in } \Omega \times (0,T), \tag{1}$$

$$b(x,u)(t=0) = b(x,u_0) \quad \text{in } \Omega, \tag{2}$$

$$u = 0 \quad \text{on } \partial\Omega \times (0,T), \tag{3}$$

where Ω is a bounded open set of $I\!R^N$, $(N \geq 1)$, T is a positive real number while the data $b(x,u_0)$ in $L^1(\Omega)$. The matrix $A(x,t)$ is a bounded symmetric and coercive matrix. The function $b(x,s)$ is assumed to strictly increasing and C^1 for s (for every $x \in \Omega$) but which is not restricted by any growth condition with respect to s (see assumptions (4) and (5) of Section 2), f is a Carathéodory function in $\Omega \times (0,T) \times I\!R$ and not controlled with respect to s. The function Φ is just assumed to be continuous on $I\!R$.

A large number of papers was devoted to the study of the existence and uniqueness for solution of parabolic problems under various assumptions and in different contexts: for a review on classical results (see, e.g. ,[2] ,[4] ,[6] ,[7] ,[10] ,[11 12]).

When Problems (1)-(3) is investigated of difficulty is due to the facts that the data f and $b(x, u_0)$ only belong to L^1, the function $f(x,t,u)$, $\Phi(u)$, does not belong $L^1_{loc}(\Omega \times (0,T))$ in general, and the main difficulty relies on the dependence of $b(x,u)$ on both x and u, so that proving existence of a weak solution (i.e. in the distribution meaning) seems to be an arduous task.

The existence of renormalized solutions (1)-(3) has been proved in H. Redwane[22] in the case where $f(x,t,u)$ is independent of u and where $-div(A(x,t)Du)$ is replaced by the operator $-div(a(x,t,u,Du))$ (the operator is a Leray-Lions which is coercive and which grows like $|Du|^{p-1}$ with respect to Du, but which is not restrict by any growth condition with respect to u), and the uniqueness of renormalized solutions has been proved in H. Redwane[23] in the case where $f(x,t,u)$ is independent of u.

2. Assumptions on the data and definition of a renormalized solution

We take Ω a bounded open set on $I\!R^N$ $(N \geq 1)$, $T > 0$ is given and we set $Q = \Omega \times (0,T)$.

$$b : \Omega \times I\!R \to I\!R \text{ is a Carathéodory function such that ;} \tag{4}$$

for every $x \in \Omega$: $b(x,s)$ is a strictly increasing C^1-function, with $b(x,0) = 0$.
For any $K > 0$, there exists $\lambda_K > 0$, a function A_K in $L^\infty(\Omega)$ and a function B_K in $L^2(\Omega)$ such that

$$\lambda_K \leq \frac{\partial b(x,s)}{\partial s} \leq A_K(x) \quad \text{and} \quad \left|\nabla_x\Big(\frac{\partial b(x,s)}{\partial s}\Big)\right| \leq B_K(x) \tag{5}$$

for almost every $x \in \Omega$, for every s such that $|s| \leq K$.

$$A(x,t) \text{ is a symmetric coercive matrix field with coefficients} \tag{6}$$

lying in $L^\infty(Q)$ i.e. $A(x,t) = (a_{ij}(x,t))_{1\leq i,\ j \leq N}$ with $a_{ij}(x,t) \in L^\infty(Q)$ and $a_{ij}(x,t) = a_{ji}(x,t)$ a.e. in Q, $\forall i,\ j$, and there exists $\alpha > 0$ such that $A(x,t)\xi.\xi \geq \alpha|\xi|^2$ a.e. in Q, $\forall \xi \in I\!R^N$.

$$\Phi \ : \ I\!R \to I\!R^N \text{ is a continuous function.} \tag{7}$$

$$f \ : \ Q \times I\!R \to I\!R \quad \text{is a Carathéodory function.} \tag{8}$$

For almost every $(x,t) \in Q$, for every $s \in I\!R$:

$$sign(s) f(x,t,s) \geq 0 \text{ and } f(x,t,0) = 0. \tag{9}$$

For any $K > 0$, there exists $\sigma_K > 0$ and a function F_K in $L^2(Q)$ such that

$$|f(x,t,s)| \leq F_K(x,t) + \sigma_K \ |s| \tag{10}$$

for almost every $(x,t) \in Q$, for every s such that $|s| \leq K$.

$$u_0 \text{ is a measurable function defined on } \Omega \text{ such that } b(x,u_0) \in L^1(\Omega). \tag{11}$$

Remark 2.1. As already mentioned in the introduction, Problems (1)-(3) does not admit a weak solution under assumptions (7)-(11) since the growths of $b(x,u)$, $\Phi(u)$ and $f(x,t,u)$ are not controlled with respect to u (so that these fields are not in general defined as distributions, even when u belongs $L^2(0,T;H_0^1(\Omega))$).

Throughout this paper and for any non negative real number K we denote by $T_K(r) = min(K, max(r,-K))$ the truncation function at height K. The definition of a renormalized solution for Problems (1)-(3) can be stated as follows.

Definition 2.1. A measurable function u defined on Q is a renormalized solution of Problems (1)-(3) if

$$T_K(u) \in L^2(0,T;H_0^1(\Omega)) \text{ for any } K \geq 0 \text{ and } b(x,u) \in L^\infty(0,T;L^1(\Omega)), \tag{12}$$

$$\int_{\{(t,x)\in Q \ ; \ n\leq|u(x,t)|\leq n+1\}} A(x,t)Du.Du \, dx \, dt \ \longrightarrow 0 \ \text{ as } n \to +\infty, \tag{13}$$

and if, for every a function S in $W^{2,\infty}(I\!R)$ increasing, which is piecewise C^1 and such that S' has a compact support, we have

$$\frac{\partial b_S(x,u)}{\partial t} - div\Big(S'(u)A(x,t)\Big) + S''(u)A(x,t)Du.Du \tag{14}$$

$$- div\Big(S'(u)\Phi(u)\Big) + S''(u)\Phi(u)Du + f(x,t,u)S'(u) = 0 \ \text{ in } D'(Q),$$

$$b_S(x,u)(t=0) = b_S(x,u_0) \ \text{ in } \Omega, \tag{15}$$

where $b_S(x,r) = \displaystyle\int_0^r \frac{\partial b(x,s)}{\partial s} S'(s)\, ds.$

The following remarks are concerned with a few comments on definition 2.1.

Remark 2.2. Equation (14) is formally obtained through pointwise multiplication of equation (1) by $S'(u)$. Note that due to (12) each term in (14) has a meaning in $L^1(Q) + L^2(0,T;H^{-1}(\Omega))$.

Indeed, if $K \geq 0$ is such that $suppS' \subset [-K, K]$, the following identifications are made in (14) :

$\star$ $b_S(x,u)$ belongs to $L^\infty(Q)$ because $|b_S(x,u)| \leq \|A_K\|_{L^\infty(\Omega)} \|S\|_{L^\infty(I\!R)}$.

$\star$ $S'(u)A(x,t)Du$ identifies with $S'(u)A(x,t)DT_K(u)$ a.e. in Q. Since indeed $|T_K(u)| \leq K$ a.e. in Q, assumptions (6) and (12) imply that

$$S'(u)A(x,t)DT_K(u) \in L^2(Q)^N.$$

$\star$ $S''(u)A(x,t)DuDu$ identifies with $S''(u)A(x,t)DT_K(u).DT_K(u)$ and in view of (6), and (12) one has

$$S''(u)A(x,t)DT_K(u).DT_K(u)) \in L^1(Q).$$

$\star$ $S'(u)\Phi(u)$ and $S''(u)\Phi(u)\nabla u$ respectively identify with $S'(u)\Phi(T_K(u))$ and $S''(u)\Phi(T_K(u))\nabla T_K(u)$. Due to the properties of S and (7), the functions S', S'' and $\Phi \circ T_K$ are bounded on $I\!R$ so that (12) implies that $S'(u)\Phi(T_K(u)) \in (L^\infty(Q))^N$, and $S''(u)\Phi(T_K(u))DT_K(u) \in L^2(Q)$.

$\star$ $f(x,t,u)S'(u)$ identifies with $f(x,t,T_K(u))S'(u)$ and in view of (10) one has $f(x,t,T_K(u))S'(u) \in L^2(Q)$.

The above considerations show that equation (14) takes place in $D'(Q)$ and that $\frac{\partial b_S(x,u)}{\partial t}$ belongs to $L^1(Q) + L^2(0,T;H^{-1}(\Omega))$. Due to the properties of S and (5), we have $b_S(x,u)$ belongs to $L^2(0,T;H_0^1(\Omega))$ (see (17)), which implies that $b_S(x,u)$ belongs to $C^0([0,T];L^1(\Omega))$ (for a proof of this trace result see [20]), so that the initial condition (15) makes sense.

Remark 2.3. Due to the properties of S (increasing) and (5), we have

$$\lambda_K|S(r)-S(r')| \leq \Big|b_S(x,r)-b_S(x,r')\Big| \leq A_K(x)|S(r)-S(r')| \quad \forall r,\ r' \in I\!R \tag{16}$$

and

$$|Db_S(x,u)| \leq \|A_K\|_{L^\infty(Q)}\|S\|_{L^\infty(I\!R)}|DT_K(u)| + K|B_K(x)|\|S'\|_{L^\infty(I\!R)}. \tag{17}$$

3. Existence result

This section is devoted to establish the following existence theorem.

Theorem 3.1. *Under assumptions (7)-(11) there exists at least a renormalized solution u of Problems (1)-(3).*

Proof of Theorem 3.1.
The proof is divided into 7 steps.

⋆ **Step 1** : **Approximate problem**. Let us introduce the following regularization of the data: for $\varepsilon > 0$ fixed

$$b_\varepsilon(x,s) = b(x, T_{\frac{1}{\varepsilon}}(s))) \quad \text{a.e. in } \Omega,\ \forall s \in I\!R. \tag{18}$$

$$\Phi_\varepsilon \text{ is a lipschitz-continuous bounded function from } I\!R \text{ into } I\!R^N \tag{19}$$

such that Φ_ε uniformly converges to Φ on any compact subset of $I\!R$ as ε tends to 0.

$$f^\varepsilon(x,t,s) = f(x,t,T_{\frac{1}{\varepsilon}}(s)) \quad \text{a.e. in } Q,\ \forall s \in I\!R. \tag{20}$$

$$u_0^\varepsilon \in C_0^\infty(\Omega) \ : \ b_\varepsilon(x,u_0^\varepsilon) \longrightarrow b(x,u_0) \text{ in } L^1(\Omega) \text{ as } \varepsilon \text{ tends to } 0. \tag{21}$$

Let us now consider the following regularized problem:

$$\frac{\partial b_\varepsilon(x,u^\varepsilon)}{\partial t} - div\Big(A(x,t)Du^\varepsilon + \Phi_\varepsilon(u^\varepsilon)Du^\varepsilon\Big) + f^\varepsilon(x,t,u^\varepsilon) = 0 \ \text{ in } Q, \tag{22}$$

$$u^\varepsilon = 0 \ \text{ on } (0,T)\times\partial\Omega, \tag{23}$$

$$b_\varepsilon(x,u^\varepsilon)(t=0) = b_\varepsilon(x,u_0^\varepsilon) \ \text{ in } \Omega. \tag{24}$$

In view of (18), b_ε satisfy (4) and (5), and due to (5), there exists $\lambda_\varepsilon > 0$, a function A_ε in $L^\infty(\Omega)$ and a function B_ε in $L^2(\Omega)$ such that

$$\lambda_\varepsilon \le \frac{\partial b_\varepsilon(x,s)}{\partial s} \le A_\varepsilon(x) \qquad \text{and} \qquad \left|\nabla_x\Big(\frac{\partial b_\varepsilon(x,s)}{\partial s}\Big)\right| \le B_\varepsilon(x) \text{ a.e. in } \Omega, \tag{25}$$

$\forall s \in I\!R$. In view of (20), f^ε satisfy (8), (9) and (10), and due to (10), there exists $\sigma_\varepsilon > 0$ and a function F_ε in $L^2(Q)$ such that

$$|f^\varepsilon(x,t,s)| \le F_\varepsilon(x,t) + \sigma_\varepsilon\ |s| \tag{26}$$

As a consequence, proving existence of a weak solution $u^\varepsilon \in L^2(0,T;H_0^1(\Omega))$ of (22)-(24) is an easy task (see e.g. [15]).

⋆ **Step 2** : **A priori estimates**. The estimates derived in this step rely on usual techniques for problems of type (22)-(24) and we just sketch the proof of them (the reader is referred to ,[2] ,[3] ,[7] ,[4] [5] or to ,[8] ,[18] [19] for elliptic versions of (22)-(24)).

Using $T_K(u^\varepsilon)$ as a test function in (22) leads to

$$\int_\Omega b_K^\varepsilon(x,u^\varepsilon)(t)\,dx + \int_0^t\int_\Omega A(x,t)Du^\varepsilon.DT_K(u^\varepsilon)\,dx\,ds \tag{27}$$

$$+\int_0^t\int_\Omega \Phi_\varepsilon(u^\varepsilon)DT_K(u^\varepsilon)\,dx\,ds+\int_0^t\int_\Omega f^\varepsilon(x,t,u^\varepsilon)T_K(u^\varepsilon)\,dx\,ds$$
$$=\int_\Omega b_K^\varepsilon(x,u_0^\varepsilon)\,dx$$

for almost every t in $(0,T)$, and where $b_K^\varepsilon(x,r)=\displaystyle\int_0^r T_K(s)\frac{\partial b_\varepsilon(x,s)}{\partial s}\,ds.$

The Lipschitz character of Φ_ε, Stokes formula together with the boundary condition (24) make it possible to obtain

$$\int_0^t\int_\Omega \Phi_\varepsilon(u^\varepsilon)DT_K(u^\varepsilon)\,dx\,ds=0, \tag{28}$$

for almost any $t\in(0,T)$.

Due to the definition of b_K^ε we have $0\le\displaystyle\int_\Omega b_K^\varepsilon(x,u_0^\varepsilon)\,dx\le K\int_\Omega |b_\varepsilon(x,u_0^\varepsilon)|\,dx.$

Observing that both terms on the left hand side of the above equality are nonnegative, and since $A(x,t)$ satisfies (6), the properties of $b_\varepsilon(x,u_0^\varepsilon)$, permit to deduce from (27) that

$$T_K(u^\varepsilon) \text{ is bounded in } L^2(0,T;H_0^1(\Omega)) \tag{29}$$

independently of ε for any $K\ge 0$.

Proceeding as in ,[3] [4] and [7] that for any $S\in W^{2,\infty}(I\!R)$ such that S' is compact ($suppS'\subset[-K,K]$)

$$b_S(x,u^\varepsilon) \text{ is bounded in } L^2(0,T;H_0^1(\Omega)) \tag{30}$$

and

$$\frac{\partial b_S(x,u^\varepsilon)}{\partial t} \text{ is bounded in } L^1(Q)+L^2(0,T;H^{-1}(\Omega)) \tag{31}$$

independently of ε.

As a consequence of (17), (25) and (29) we then obtain (30). To show that (31) holds true, we multiply the equation for u^ε in (22) by $S'(u^\varepsilon)$ to obtain

$$\frac{\partial b_S^\varepsilon(x,u^\varepsilon)}{\partial t}=div\Big(S'(u^\varepsilon)A(x,t)Du^\varepsilon\Big) \tag{32}$$

$$-S''(u^\varepsilon)A(x,t)Du^\varepsilon.Du^\varepsilon+\ div(\Phi_\varepsilon(u^\varepsilon))S'(u^\varepsilon)-f^\varepsilon(x,t,u^\varepsilon)S'(u^\varepsilon)=0$$

in $D'(Q)$, where $b_S^\varepsilon(x,r)=\displaystyle\int_0^r\frac{\partial b^\varepsilon(x,s)}{\partial s}S'(s)\,ds.$

Since $suppS'$ and $suppS''$ are both included in $[-K, K]$, u^ε may be replaced by $T_K(u^\varepsilon)$ in each of these terms. As a consequence, each term in the right hand side of (32) is bounded either in $L^2(0,T;H^{-1}(\Omega))$ or in $L^1(Q)$. (see ,[4] [7]). As a consequence of (6), (7), (10) and (29) we then obtain (31).

For any integer $n \geq 1$, consider the Lipschitz-continuous function θ_n defined through $\theta_n(r) = T_{n+1}(r) - T_n(r)$. Remark that $\|\theta_n\|_{L^\infty(I\!R)} \leq 1$ for any $n \geq 1$ and that $\theta_n(r) \to 0$ for any r when n tends to infinity.

Using that admissible test function $\theta_n(u^\varepsilon)$ in (22) leads to

$$\int_\Omega b_{\varepsilon,n}(x,u^\varepsilon)(t)\,dx + \int_0^t \int_\Omega A(x,t)Du^\varepsilon.D\theta_n(u^\varepsilon)\,dx\,ds \tag{33}$$

$$+\int_0^t \int_\Omega \Phi_\varepsilon(u^\varepsilon)D\theta_n(u^\varepsilon)\,dx\,ds = \int_0^t \int_\Omega f^\varepsilon(x,u^\varepsilon)\theta_n(u^\varepsilon)\,dx\,ds$$
$$= \int_\Omega b_{\varepsilon,n}(x,u_0^\varepsilon)\,dx,$$

for almost any t in $(0,T)$ and where $b_{\varepsilon,n}(x,r) = \displaystyle\int_0^r \frac{\partial b_\varepsilon(x,s)}{\partial s}\theta_n(s)\,ds.$

The Lipschitz character of Φ_ε, and since

$$b_{\varepsilon,n}(x,r) \geq 0,\ \ f^\varepsilon(x,t,u^\varepsilon)\theta_n(u^\varepsilon),$$

equality (33) implies that

$$\int_0^t \int_\Omega A(x,t)Du^\varepsilon.D\theta_n(u^\varepsilon)\,dx\,ds \leq \int_\Omega b_{\varepsilon,n}(x,u_0^\varepsilon)\,dx, \tag{34}$$

for almost $t \in (0,T)$.

$\star$ **Step 3 : Limit of the approximate solutions**. Arguing again as in ,[3] ,[4] [5] and [7] estimates (30) and (31) imply that, for a subsequence still indexed by ε,

$$b_\varepsilon(x,u^\varepsilon) \text{ converges strongly in } L^1(Q) \text{ and almost every where to } b(x,u) \tag{35}$$

in Q and with the help of (16) and (29),

$$u^\varepsilon \text{ converges almost every where to } u \text{ in } Q, \tag{36}$$

$$T_K(u^\varepsilon) \text{ converges weakly to } T_K(u) \text{ in } L^2(0,T;H_0^1(\Omega)), \tag{37}$$

$$\theta_n(u^\varepsilon) \rightharpoonup \theta_n(u) \text{ weakly in } L^2(0,T;H_0^1(\Omega)) \tag{38}$$

as ε tends to 0 for any $K > 0$ and any $n \geq 1$.

We now establish that $b(x,u)$ belongs to $L^\infty(0,T;L^1(\Omega))$. Indeed using $\frac{1}{\sigma}T_\sigma(u^\varepsilon)$ as a test function in (22) and letting σ go to zero, it follows that

$$\int_\Omega |b_\varepsilon(x,u^\varepsilon)|(t)\,dx \leq \|b_\varepsilon(x,u_0^\varepsilon)\|_{L^1(\Omega)} \quad \text{a.e. in } (0,T). \tag{39}$$

With of (21) and (35), we have $b(x,u)$ belongs to $L^\infty(0,T;L^1(\Omega))$.

We are now in a position to exploit (34). Due to the definition of θ_n, the pointwise convergence of u^ε to u and $b_\varepsilon(x,u_0^\varepsilon)$ to $b(x,u_0)$ then imply that

$$\overline{\lim_{\varepsilon\to 0}} \int_{\{n\leq|u^\varepsilon|\leq n+1\}} A(x,t)Du^\varepsilon.Du^\varepsilon\,dx\,dt \leq \int_\Omega b_n(x,u_0)\,dx.$$

Since θ_n converge to zero everywhere as n goes to zero. The Lebesgue's convergence theorem permits to conclude that

$$\lim_{n\to+\infty}\overline{\lim_{\varepsilon\to 0}} \int_{\{n\leq|u^\varepsilon|\leq n+1\}} A(x,t)DT_{n+1}(u^\varepsilon).DT_{n+1}(u^\varepsilon)\,dx\,dt = 0. \tag{40}$$

Since $A(x,t)DT_{n+1}(u^\varepsilon).DT_{n+1}(u^\varepsilon) = \left|A(x,t)^{\frac{1}{2}}DT_{n+1}(u^\varepsilon)\right|^2$ (with $A^{\frac{1}{2}}$ denotes the coercive symmetric matrix such that $A^{\frac{1}{2}}.A^{\frac{1}{2}} = A$) and (37), (6) imply that $A(x,t)^{\frac{1}{2}}DT_{n+1}(u^\varepsilon) \rightharpoonup A(x,t)^{\frac{1}{2}}DT_{n+1}(u)$ weakly in $(L^2(Q))^N$, and (13) is then established.

$\star$ **Step 4** : **Time regularization**. This step is devoted to introduce for $K\geq 0$ fixed, a time regularization of the function $T_K(u)$ in order to perform the monotonicity method which will be developed in Step 5 and Step 6. This kind of regularization has been first introduced by R. Landes (see Lemma 6 and Proposition 3, p. 230 and Proposition 4, p. 231 in [14]). More recently, it has been exploited in [5] and [13] to solve a few nonlinear evolution problems with L^1 or measure data.

This specific time regularization of $T_K(u)$ (for fixed $K\geq 0$) is defined as follows. Let $(v_0^\mu)_\mu$ be a sequence of functions defined on Ω such that

$$v_0^\mu \in L^\infty(\Omega)\cap H_0^1(\Omega) \text{ for all } \mu>0, \tag{41}$$

$$\|v_0^\mu\|_{L^\infty(\Omega)} \leq K \quad \forall\mu>0, \tag{42}$$

$$v_0^\mu \to T_K(u_0) \text{ a.e. in } \Omega \text{ and } \frac{1}{\mu}\|v_0^\mu\|_{L^2(\Omega)} \to 0, \text{ as } \mu\to+\infty. \tag{43}$$

Existence of such a subsequence $(v_0^\mu)_\mu$ is easy to establish (see e.g. [21]). For fixed $K\geq 0$ and $\mu>0$, let us consider the unique solution $T_K(u)_\mu \in$

$L^\infty(Q) \cap L^2(0,T;H_0^1(\Omega))$ of the monotone problem:

$$\frac{\partial T_K(u)_\mu}{\partial t} + \mu\Big(T_K(u)_\mu - T_K(u)\Big) = 0 \text{ in } D'(Q). \tag{44}$$

$$T_K(u)_\mu(t=0) = v_0^\mu \text{ in } \Omega. \tag{45}$$

Remark that due to (44), we have for $\mu > 0$ and $K \geq 0$,

$$\frac{\partial T_K(u)_\mu}{\partial t} \in L^2(0,T;H_0^1(\Omega)). \tag{46}$$

The behavior of $T_K(u)_\mu$ as $\mu \to +\infty$ is investigated in [14] (see also ,[5] [13] and [21]) and we just recall here that (44)-(45) imply that

$$T_K(u)_\mu \to T_K(u) \;\text{ a.e. in } Q\ ; \tag{47}$$

and in $L^\infty(Q)$ weak $\star$ and strongly in $L^2(0,T;H_0^1(\Omega))$ as $\mu \to +\infty$.

$$\|T_K(u)_\mu\|_{L^\infty(Q)} \leq max\Big(\|T_K(u)\|_{L^\infty(Q)}\ ;\ \|v_0^\mu\|_{L^\infty(\Omega)}\Big) \leq K \tag{48}$$

for any μ and any $K \geq 0$.

Let $h \in W^{1,\infty}(I\!R),\ h \geq 0$, supp h is compact. The main estimate is

Lemma 3.1.

$$\underline{\lim_{\mu\to+\infty}} \lim_{\varepsilon\to 0} \int_0^T \int_0^s \Big\langle \frac{\partial b_\varepsilon(x,u^\varepsilon)}{\partial t}\ , h(u^\varepsilon)\Big(T_K(u^\varepsilon) - (T_K(u))_\mu\Big)\Big\rangle\, dt\, ds \geq 0$$

where $\langle\ ,\ \rangle$ denotes the duality pairing between $L^1(\Omega) + H^{-1}(\Omega)$ and $L^\infty(\Omega) \cap H_0^1(\Omega)$.

Proof of Lemma 3.1 : The Lemma is proved in .[22]

$\star$ **Step 5**. In this step we prove the following lemma which is the key point in the monotonocity arguments that will be developed in Step 6.

Lemma 3.2. *The subsequence of u^ε defined is Step 3 satisfies for any $K \geq 0$*

$$\overline{\lim_{\varepsilon\to 0}} \int_0^T \int_0^t \int_\Omega A(x,t) DT_K(u^\varepsilon).DT_K(u^\varepsilon)\, dx\, ds\, dt \tag{49}$$

$$\leq \int_0^T \int_0^t \int_\Omega A(x,t) DT_K(u).DT_K(u)\, dx\, ds\, dt.$$

Proof of Lemma 3.2: We first introduce a sequence of increasing $C^\infty(I\!R)$-functions S_n such that, for any $n \geq 1$, $S_n(r) = r$ for $|r| \leq n$, $supp(S'_n) \subset [-(n+1),(n+1)]$ and $\|S''_n\|_{L^\infty(I\!R)} \leq 1$.

We use the sequence $T_K(u)_\mu$ of approximations of $T_K(u)$ defined by (44), (45) of Step 4, and plug the test function $S'_n(u^\varepsilon)(T_K(u^\varepsilon) - T_K(u)_\mu)$ (for $\varepsilon > 0$ and $\mu > 0$) in (22). Through setting, for fixed $K \geq 0$,

$$W_\mu^\varepsilon = (T_K(u^\varepsilon) - T_K(u)_\mu) \tag{50}$$

we obtain upon integration over $(0,t)$ and then over $(0,T)$:

$$\begin{aligned}
&\int_0^T \int_0^t \left\langle \frac{\partial b_\varepsilon(x,u^\varepsilon)}{\partial t} \ , \ S'_n(u^\varepsilon) W_\mu^\varepsilon \right\rangle ds\, dt \\
&+ \int_0^T \int_0^t \int_\Omega S'_n(u^\varepsilon) A(x,t) Du^\varepsilon . DW_\mu^\varepsilon \, dx\, ds\, dt \\
&+ \int_0^T \int_0^t \int_\Omega S''_n(u^\varepsilon) W_\mu^\varepsilon A(x,t) Du^\varepsilon . Du^\varepsilon \, dx\, ds\, dt \\
&+ \int_0^T \int_0^t \int_\Omega \Phi_\varepsilon(u^\varepsilon) S'_n(u^\varepsilon) DW_\mu^\varepsilon \, dx\, ds\, dt \\
&+ \int_0^T \int_0^t \int_\Omega S''_n(u^\varepsilon) W_\mu^\varepsilon \Phi_\varepsilon(u^\varepsilon) Du^\varepsilon \, dx\, ds\, dt \\
&+ \int_0^T \int_0^t \int_\Omega f^\varepsilon(x,t,u^\varepsilon) S'_n(u^\varepsilon) W_\mu^\varepsilon \, dx\, ds\, dt = 0.
\end{aligned} \tag{51}$$

In the following we pass to the limit in (51) as ε tends to 0, then μ tends to $+\infty$ and then n tends to $+\infty$, the real number $K \geq 0$ being kept fixed. In order to perform this task we prove below the following results for fixed $K \geq 0$:

$$\underline{\lim}_{\mu\to+\infty} \lim_{\varepsilon\to 0} \int_0^T \int_0^t \left\langle \frac{\partial b_\varepsilon(x,u^\varepsilon)}{\partial t} \ , \ S'_n(u^\varepsilon) W_\mu^\varepsilon \right\rangle ds\, dt \geq 0 \ \text{ for any } n \geq K, \tag{52}$$

$$\lim_{\mu\to+\infty} \lim_{\varepsilon\to 0} \int_0^T \int_0^t \int_\Omega S'_n(u^\varepsilon) \Phi_\varepsilon(u^\varepsilon) DW_\mu^\varepsilon \, dx\, ds\, dt = 0 \ \text{ for any } n \geq 1, \tag{53}$$

$$\lim_{\mu\to+\infty} \lim_{\varepsilon\to 0} \int_0^T \int_0^t \int_\Omega S''_n(u^\varepsilon) W_\mu^\varepsilon \Phi_\varepsilon(u^\varepsilon) Du^\varepsilon \, dx\, ds\, dt = 0 \ \text{ for any } n, \tag{54}$$

$$\lim_{n\to+\infty}\overline{\lim_{\mu\to+\infty}}\ \overline{\lim_{\varepsilon\to 0}}\left|\int_0^T\int_0^t\int_\Omega S_n''(u^\varepsilon)W_\mu^\varepsilon A(x,t)Du^\varepsilon.Du^\varepsilon\,dx\,ds\,dt\right|=0, \tag{55}$$

and

$$\lim_{\mu\to+\infty}\lim_{\varepsilon\to 0}\int_0^T\int_0^t\int_\Omega f^\varepsilon S_n'(u^\varepsilon)W_\mu^\varepsilon\,dx\,ds\,dt=0 \quad\text{for any } n\geq 1. \tag{56}$$

Proof of (52). In view of the definition (50) of W_μ^ε, lemma 3.1 applies with $h=S_n$ for fixed $n\geq K$. As a consequence (52) holds true.

Proof of (53). For fixed $n\geq 1$, we have

$$S_n'(u^\varepsilon)\Phi_\varepsilon(u^\varepsilon)DW_\mu^\varepsilon=S_n'(u^\varepsilon)\Phi_\varepsilon(T_{n+1}(u^\varepsilon))DW_\mu^\varepsilon \tag{57}$$

a.e. in Q, and where $suppS_n'\subset[-(n+1),n+1]$. Since S_n' is smooth and bounded, (19) and (36) lead to $S_n'(u^\varepsilon)\Phi_\varepsilon(T_{n+1}(u^\varepsilon))$ converges to $S_n'(u)\Phi(T_{n+1}(u))$ a.e. in Q and in $L^\infty(Q)$ weak $\star$, as ε tends to 0. For fixed $\mu>0$, we have

$$W_\mu^\varepsilon\rightharpoonup(T_K(u)-T_K(u)_\mu)\ \text{ weakly } in\ L^2(0,T;H_0^1(\Omega)) \tag{58}$$

and a.e. in Q and in $L^\infty(Q)$ weak $\star$, as ε tends to 0. As a consequence of (57) and (58) we deduce that

$$\lim_{\varepsilon\to 0}\int_0^T\int_0^t\int_\Omega S_n'(u^\varepsilon)\Phi_\varepsilon(u^\varepsilon)DW_\mu^\varepsilon\,dx\,ds\,dt \tag{59}$$

$$=\int_0^T\int_0^t\int_\Omega S_n'(u)\Phi(u)D\Big[T_K(u)-T_K(u)_\mu\Big]\,dx\,ds\,dt$$

for any $\mu>0$.

Appealing now to (47) and passing to the limit as $\mu\to+\infty$ in (59) allows to conclude that (53) holds true.

Proof of (54). For fixed $n\geq 1$, and by the same arguments that those that lead to (53), we have

$$S_n''(u^\varepsilon)\Phi_\varepsilon(u^\varepsilon)Du^\varepsilon W_\mu^\varepsilon=S_n''(u^\varepsilon)\Phi_\varepsilon(T_{n+1}(u^\varepsilon))DT_{n+1}(u^\varepsilon)W_\mu^\varepsilon \text{ a.e. in } Q.$$

From (19) and (36), it follows that for any $\mu>0$

$$\lim_{\varepsilon\to 0}\int_0^T\int_0^t\int_\Omega S_n''(u^\varepsilon)\Phi_\varepsilon(u^\varepsilon)Du^\varepsilon W_\mu^\varepsilon\,dx\,ds\,dt$$

$$=\int_0^T\int_0^t\int_\Omega S_n''(u^\varepsilon)\Phi_\varepsilon(T_{n+1}(u^\varepsilon))DT_{n+1}(u^\varepsilon)W_\mu^\varepsilon\,dx\,ds\,dt$$

with the help of (58) passing to the limit, as μ tends to $+\infty$, in the above equality leads to (53).

Proof of (55). For any $n \geq 1$ fixed, we have $suppS''_n \subset [-(n+1), -n] \cup [n, n+1]$. As a consequence

$$\Big|\int_0^T \int_0^t \int_\Omega S''_n(u^\varepsilon)A(x,t)Du^\varepsilon.Du^\varepsilon W^\varepsilon_\mu \, dx \, ds \, dt\Big|$$

$$\leq TC \int_{\{n \leq |u^\varepsilon| \leq n+1\}} A(x,t)Du^\varepsilon.Du^\varepsilon \, dx \, dt,$$

for any $n \geq 1$, and any $\mu > 0$, where C is a constant independent of n, μ. With the help of (40) passing to the limit, as ε tends to zero, μ tends to $+\infty$ and n tends to $+\infty$ and to establish (55).

Proof of (56). For fixed $n \geq 1$, and in view (20), (36) and (58), Lebesgue's convergence theorem implies that for any $\mu > 0$ and any $n \geq 1$

$$\lim_{\varepsilon \to 0} \int_0^T \int_0^t \int_\Omega f^\varepsilon(x,t,T_{n+1}(u^\varepsilon))S'_n(u^\varepsilon)W^\varepsilon_\mu \, dx \, ds \, dt$$

$$= \int_0^T \int_0^t \int_\Omega f(x,t,T_{n+1}(u))S'_n(u)\Big(T_K(u) - T_K(u)_\mu\Big) \, dx \, ds \, dt.$$

Now for fixed $n \geq 1$, using (47) permits to pass to the limit as μ tends to $+\infty$ in the above equality to obtain (56).

We now turn back to the proof of lemma 3.2, due to (51), (52), (53), (54), (55) and (56), we are in a position to pass to the lim-sup when ε tends to zero, then to the limit-sup when μ tends to $+\infty$ and then to the limit as n tends to $+\infty$ in (51). We obtain using the definition of W^ε_μ that for any $K \geq 0$

$$\lim_{n \to +\infty} \overline{\lim_{\mu \to +\infty} \lim_{\varepsilon \to 0}} \int_0^T \int_0^t \int_\Omega S'_n(u^\varepsilon)A(x,t) \times Du^\varepsilon D\Big(T_K(u^\varepsilon) - T_K(u)_\mu\Big) \, dx \, ds \, dt \leq 0.$$

Since $S'_n(u^\varepsilon)DT_K(u^\varepsilon) = DT_K(u^\varepsilon)$ for $K \leq n$.

The above inequality implies that for $K \leq n$

$$\overline{\lim_{\varepsilon \to 0}} \int_0^T \int_0^t \int_\Omega A(x,t)Du^\varepsilon.DT_K(u^\varepsilon) \, dx \, ds \, dt \tag{60}$$

$$\leq \lim_{n \to +\infty} \overline{\lim_{\mu \to +\infty}} \; \overline{\lim_{\varepsilon \to 0}} \int_0^T \int_0^t \int_\Omega S'_n(u^\varepsilon)A(x,t)DT_{n+1}(u^\varepsilon).DT_K(u)_\mu \, dx \, ds \, dt.$$

The right hand side of (61) is computed as follows. Due to (37) it follows that for fixed $n \geq 1$

$$S'_n(u^\varepsilon)DT_{n+1}(u^\varepsilon) \rightharpoonup S'_n(u)DT_{n+1}(u) \text{ weakly in } (L^2(Q))^N$$

when ε tends to 0. The strong convergence of $T_K(u)_\mu$ to $T_K(u)$ in $L^2(0,T;H_0^1(\Omega))$ as μ tends to $+\infty$, then allows to conclude that

$$\lim_{\mu\to+\infty}\lim_{\varepsilon\to 0}\int_0^T\int_0^t\int_\Omega S'_n(u^\varepsilon)A(x,t)DT_{n+1}(u^\varepsilon).DT_K(u)_\mu\,dx\,ds\,dt \tag{61}$$

$$=\int_0^T\int_0^t\int_\Omega S'_n(u)A(x,t)DT_{n+1}(u).DT_K(u)\,dx\,ds\,dt$$

$$=\int_0^T\int_0^t\int_\Omega A(x,t)DT_K(u).DT_K(u)\,dx\,ds\,dt$$

as soon as $K \leq n$, since $S'_n(r) = 1$ for $|r| \leq n$.

Recalling (60) and (61) allows to conclude (49) holds true and the proof of lemma 3.2 is complete.

$\star$ **Step 6** : **The strong convergence of truncates**. In this step we prove the following monotonicity estimate :

Lemma 3.3. *The subsequence of u^ε defined in step 3 satisfies for any $K \geq 0$*

$$\lim_{\varepsilon\to 0}\int_0^T\int_0^t\int_\Omega A(x,t)[DT_K(u^\varepsilon)-DT_K(u)].[DT_K(u^\varepsilon)-DT_K(u)]dx\,dt\,ds = 0. \tag{62}$$

And

$$T_K(u^\varepsilon) \longrightarrow T_K(u) \text{ strongly in } L^2(0,T,H_0^1(\Omega)) \tag{63}$$

as ε goes to zero.

Proof of Lemma 3.3. Let $K \geq 0$ be fixed. we have

$$\int_0^T\int_0^t\int_\Omega A(x,t)\Big[DT_K(u^\varepsilon)-DT_K(u)\Big].\Big[DT_K(u^\varepsilon)-DT_K(u)\Big]\,dx\,dt\,ds \tag{64}$$

$$=\int_0^T\int_0^t\int_\Omega A(x,t)DT_K(u^\varepsilon).DT_K(u^\varepsilon)\,dx\,dt\,ds$$

$$-\int_0^T\int_0^t\int_\Omega A(x,t)DT_K(u)DT_K(u^\varepsilon)\,dx\,dt\,ds$$

$$-\int_0^T\int_0^t\int_\Omega A(x,t)\Big[DT_K(u^\varepsilon)-DT_K(u)\Big].DT_K(u)\,dx\,dt\,ds.$$

Using (49) of lemma 3.2, we obtain

$$\overline{\lim_{\varepsilon\to 0}}\leq\int_0^T\int_0^t\int_\Omega A(x,t)DT_K(u^\varepsilon).DT_K(u^\varepsilon)\,dx\,dt\,ds \tag{65}$$

$$\leq\int_0^T\int_0^t\int_\Omega A(x,t)DT_K(u).DT_K(u)\,dx\,dt\,ds.$$

As a consequence of (37) we have for all $K>0$

$$\lim_{\varepsilon\to 0}\int_0^T\int_0^t\int_\Omega A(x,t)\Big[DT_K(u^\varepsilon)-DT_K(u)\Big].DT_K(u)\,dx\,dt\,ds=0 \tag{66}$$

(65) and (66) allow to pass to the lim-sup as ε tends to zero in (64) and to obtain (62) and (63) of lemma 3.3.

$\star$ **Step 7**. In this step, u is shown to satisfies (14) and (15). Let S be a function in $W^{2,\infty}(I\!R)$ such that S' has a compact support. Let K be a positive real number such that $suppS'\subset[-K,K]$. Pointwise multiplication of the approximate equation (22) by $S'(u^\varepsilon)$ leads to

$$\frac{\partial b_S^\varepsilon(x,u^\varepsilon)}{\partial t}-div\Big(S'(u^\varepsilon)A(x,t)Du^\varepsilon\Big)+S''(u^\varepsilon)A(x,t)Du^\varepsilon.Du^\varepsilon \tag{67}$$

$$-div\Big(S'(u^\varepsilon)\Phi_\varepsilon(u^\varepsilon)\Big)+S''(u^\varepsilon)\Phi_\varepsilon(u^\varepsilon)Du^\varepsilon+f^\varepsilon(x,t,u^\varepsilon)S'(u^\varepsilon)=0\ \text{ in } D'(Q).$$

where $b_S^\varepsilon(x,r)=\displaystyle\int_0^r\frac{\partial b_\varepsilon(x,s)}{\partial s}S'(s)\,ds$. In what follows we pass to the limit as ε tends to 0 in each term of (67).

$\star$ Since S is bounded, and $b_S^\varepsilon(x,u^\varepsilon)$ converges to $b_S(x,u)$ a.e. in Q and in $L^\infty(Q)$ weak $\star$. Then $\frac{\partial b_S^\varepsilon(x,u^\varepsilon)}{\partial t}$ converges to $\frac{\partial b_S(x,u)}{\partial t}$ in $D'(Q)$ as ε tends to 0.

$\star$ Since $suppS'\subset[-K,K]$, we have for$S'(u^\varepsilon)A(x,t)Du^\varepsilon=S'(u^\varepsilon)A(x,t)DT_K(u^\varepsilon)$ a.e. in Q. The pointwise convergence of u^ε to u as ε tends to 0, the bounded character of S' and (37) imply that

$$S'(u^\varepsilon)A(x,t)DT_K(u^\varepsilon)\rightharpoonup S'(u)A(x,t)DT_K(u)\text{ weakly in }(L^2(Q))^N,$$

as ε tends to 0. And the term $S'(u)A(x,t)DT_K(u)=S'(u)A(x,t)Du$ a.e. in Q.

$\star$ Since $suppS'' \subset [-K,K]$, we have $S''(u^\varepsilon)A(x,t)Du^\varepsilon.Du^\varepsilon = A(x,t)DS'(u^\varepsilon).DT_K(u^\varepsilon)$ a.e. in Q. Due to (29) and (36), $DS'(u^\varepsilon)$ converges to $DS'(u)$ weakly in $(L^2(Q))^N$ as ε tends to 0, and (63) of lemma 3.3 allow to conclude that

$$A(x,t)DS'(u^\varepsilon).DT_K(u^\varepsilon) \rightharpoonup A(x,t)DS'(u).DT_K(u) \text{ weakly in } L^1(Q),$$

as ε tends to 0. And $A(x,t)DS'(u).DT_K(u) = S''(u)A(x,t)Du.Du$ a.e. in Q.

$\star$ Since $suppS' \subset [-K,K]$, we have $S'(u^\varepsilon)\Phi_\varepsilon(u^\varepsilon) = S'(u^\varepsilon)\Phi_\varepsilon(T_K(u^\varepsilon))$ a.e. in Q. As a consequence of (19) and (36), it follows that for any $1 \le q < +\infty$, $S'(u^\varepsilon)\Phi_\varepsilon(u^\varepsilon)$ converges to $S'(u)\Phi(T_K(u))$ strongly in $L^q(Q)$, as ε tends to 0. The term $S'(u)\Phi(T_K(u))$ is denoted by $S'(u)\Phi(u)$.

$\star$ Since $S' \in W^{1,\infty}(I\!R)$ with $suppS' \subset [-K,K]$, we have $S''(u^\varepsilon)\Phi_\varepsilon(u^\varepsilon)Du^\varepsilon = \Phi_\varepsilon(T_K(u^\varepsilon))DS'(u^\varepsilon)$ a.e. in Q, we have, $DS'(u^\varepsilon)$ converges to $DS'(u)$ weakly in $L^2(Q)^N$ as ε tends to 0, while $\Phi_\varepsilon(T_K(u^\varepsilon))$ is uniformly bounded with respect to ε and converges a.e. in Q to $\Phi(T_K(u))$ as ε tends to 0. Therefore

$$S''(u^\varepsilon)\Phi_\varepsilon(u^\varepsilon)Du^\varepsilon \rightharpoonup \Phi(T_K(u))DS'(u) \text{ weakly in } L^2(Q).$$

$\star$ Due to (20) and (35), we have $f^\varepsilon(x,t,T_{n+1}(u^\varepsilon))S'(u^\varepsilon)$ converges to $f(x,t,T_{n+1}(u))S'(u)$ strongly in $L^1(Q)$, as ε tends to 0.

As a consequence of the above convergence result, we are in a position to pass to the limit as ε tends to 0 in equation (67) and to conclude that u satisfies (28).

It remains to show that $b_S(x,u)$ satisfies the initial condition (29). To this end, firstly remark that, $S \in W^{2,\infty}(I\!R)$ such that S' has a compact support, as a consequence of (17) we have $b_S(x,u^\varepsilon)$ is bounded in $L^2(0,T;H_0^1(\Omega))$. Secondly, (67) and the above considerations on the behavior of the terms of this equation show that $\dfrac{\partial b_S^\varepsilon(x,u^\varepsilon)}{\partial t}$ is bounded in $L^1(Q)+L^2(0,T;H^{-1}(\Omega))$. As a consequence, an Aubin's type lemma (see, e.g, ,[24] Corollary 4) implies that $b_S(x,u^\varepsilon)$ lies in a compact set of $C^0([0,T];W^{-1,s}(\Omega))$ for any $s < inf\left(2, \frac{N}{N-1}\right)$. It follows that, on one hand, $b_S(x,u^\varepsilon)(t=0) = b_S(x,u_0^\varepsilon)$ converges to $b_S(x,u)(t=0)$ strongly in $W^{-1,s}(\Omega)$. On the other hand, (21) and the smoothness of S imply that $b_S(x,u_0^\varepsilon)$ converges to $b_S(x,u)(t=0)$ strongly in $L^q(\Omega)$ for all $q < +\infty$.Then we conclude that $b_S(x,u)(t=0) = b_S(x,u_0)$ in Ω. As a conclusion of step 1-step 7, the proof of theorem 3.1 is complete.

4. Comparison principle and uniqueness result

This section is concerned with a comparison principle (and a uniqueness result) for renormalized solutions in the case where $f(x,t,u)$ is independent of u. We establish the following theorem.

Theorem 4.1. *Assume that assumptions (4), (5), (6), (7) and (11) hold true and moreover that*

For any $K > 0$, there exists a positive real number $\beta_K > 0$, such that

$$\Big|\frac{\partial b(x,z_1)}{\partial s} - \frac{\partial b(x,z_2)}{\partial s}\Big| \leq \beta_K \,\Big|z_1 - z_2\Big| \tag{68}$$

for almost every x in Ω, and for every z_1 and every z_2 such that $|z_1| \leq K$ and $|z_2| \leq K$.

$$\Phi \text{ is a locally lipschitz-continuous function on } I\!R. \tag{69}$$

Let then u_1 and u_2 be renormalized solutions corresponding to the data (f_1, u_0^1) and (f_2, u_0^2) for problem ($i = 1, 2$)

$$\frac{\partial b(x,u_i)}{\partial t} - div\Big(A(x,t)Du_i + \Phi(u_i)\Big) = f_i(x,t) \quad \text{in } \Omega \times (0,T), \tag{70}$$

$$b(x,u_i)(t=0) = b(x,u_0^i) \quad \text{in } \Omega, \tag{71}$$

$$u_i = 0 \quad \text{on } \partial\Omega \times (0,T), \tag{72}$$

$$f_1,\ f_2 \in L^1(\Omega \times (0,T)). \tag{73}$$

If these data satisfying $f_1 \leq f_2$ and $u_0^1 \leq u_0^2$ almost every where, we have

$$u_1 \leq u_2 \textit{ almost every where.}$$

Sketch of the Proof of theorem 4.1. Here we give just an idea on how $u_1 \leq u_2$ can be obtained following the outlines of .[23]

The proof is divided into two Steps. In Step 1, we define a smooth approximation S_n of T_n, and we consider tow renormalized solutions u_1 and u_2 of (70)-(72) for the data (f_1, u_0^1) and (f_2, u_0^2) respectively we plug the test function $\frac{1}{\sigma}T_\sigma{}^+\Big(b_{S_n}(x,u_1) - b_{S_n}(x,u_2))\Big)$ in the difference of equations (14) for u_1 and u_2 in which we have taken $S = S_n$.

In Step 2, we investigate the behavior of the different terms in the estimate obtained in step 1 (estimates (76)) as σ tends to 0 and when n tends to $+\infty$.

⋆ *Step 1*. Remark that when Φ is locally-continuous on $I\!R$ the following derivation is licit for any function S and u satisfying the conditions mentioned in Definition 2.1.

$$div\Big(S'(u)\Phi(u)\Big) - S''(u)\Phi(u)Du = S'(u)\Phi'(u)Du = div(\Phi_S(u)) \qquad (74)$$

Where $\Phi_S = (\Phi_{S,1}, \Phi_{S,2}, \cdots, \Phi_{S,N})$ with

$$\Phi_{S,i}(r) = \int_0^r \Phi'_{S,i}(t)S'(t)\,dt.$$

Let us now introduce a specific choice of function S in (14). For all $n > 0$, let $S_n \in C^1(I\!R)$ be the function defined by $S'_n(r) = 1$ for $|r| \leq n$; $S'_n(r) = n+1-|r|$ for $n \leq |r| \leq n+1$ and $S'_n(r) = 0$ for $|r| \geq n+1$. It yields, taking $S = S_n$ in (14)

$$\frac{\partial b_{S_n}(x,u_i)}{\partial t} - \ div\Big(S'(u_i)A(x,t)Du_i\Big) + S''(u_i)A(x,t)Du_iDu_i \qquad (75)$$

$$- \ div\Big(\Phi_{S_n}(u_i)\Big) = f_iS'_n(u_i) \ \text{ in } D'(Q) \ ;$$

for $i = 1, 2$ and where $b_{S_n}(x,r) = \displaystyle\int_0^r \frac{\partial b(x,s)}{\partial s} S'_n(s)\,ds.$

We use $\frac{1}{\sigma}T^+_\sigma\Big(b_{S_n}(x,u_1) - b_{S_n}(x,u_2)\Big)$ as a test function in the difference of equations (75) for u_1 and u_2.

$$\frac{1}{\sigma}\int_0^T\int_0^t \langle \frac{\partial\Big(b_{S_n}(x,u_1) - b_{S_n}(x,u_2)\Big)}{\partial t} \ ; \ T^+_\sigma\Big(b_{S_n}(x,u_1) - b_{S_n}(x,u_2)\Big)\rangle\, ds\, dt \qquad (76)$$

$$+A^\sigma_n = B^\sigma_n + C^\sigma_n + D^\sigma_n$$

for any $\sigma > 0, \ n > 0$, and where

$$A^\sigma_n = \frac{1}{\sigma}\int_0^T\int_0^t\int_\Omega \Big[S'_n(u_1)A(t,x)Du_1 - S'_n(u_2)A(t,x)Du_2\Big] \qquad (77)$$

$$DT^+_\sigma\Big(b_{S_n}(x,u_1) - b_{S_n}(x,u_2)\Big)\, dx\, ds\, dt$$

$$B^\sigma_n = \frac{1}{\sigma}\int_0^T\int_0^t\int_\Omega S''_n(u_1)A(x,t)Du_1Du_1T^+_\sigma\Big(b_{S_n}(x,u_1) - b_{S_n}(x,u_2)\Big)\, dx\, ds\, dt \qquad (78)$$

$$-\frac{1}{\sigma}\int_0^T\int_0^t\int_\Omega S''_n(u_2)A(x,t)Du_2Du_2T^+_\sigma\Big(b_{S_n}(x,u_1) - b_{S_n}(x,u_2)\Big)\, dx\, ds\, dt$$

$$C_n^\sigma = \frac{1}{\sigma}\int_0^T\int_0^t\int_\Omega \Big[\Phi_{S_n}(u_1) - \Phi_{S_n}(u_2)\Big] DT_\sigma^+\Big(b_{S_n}(x,u_1) - b_{S_n}(x,u_2)\Big)\,dx\,ds\,dt \tag{79}$$

$$D_n^\sigma = \frac{1}{\sigma}\int_0^T\int_0^t\int_\Omega \Big[f_1 S_n'(u_1) - f_2 S_n'(u_2)\Big] T_\sigma^+\Big(b_{S_n}(x,u_1) - b_{S_n}(x,u_2)\Big)\,dx\,ds\,dt. \tag{80}$$

In the sequel we pass to the limit in (76) when σ tends to 0 and then n tends to $+\infty$. Upon application of lemma 2.4 of ,[9] the first term in the right hand side of (76) is derived as

$$\frac{1}{\sigma}\int_0^T\int_0^t \langle \frac{\partial\Big(b_{S_n}(x,u_1) - b_{S_n}(x,u_2)\Big)}{\partial t}\ ;\ T_\sigma^+\Big(b_{S_n}(x,u_1) - b_{S_n}(x,u_2)\Big)\rangle\,ds\,dt \tag{81}$$

$$\begin{aligned}&= \tfrac{1}{\sigma}\textstyle\int_Q \tilde{T}_\sigma^+\Big(b_{S_n}(x,u_1) - b_{S_n}(x,u_2)\Big)\,dx\,dt\\ &-\tfrac{T}{\sigma}\textstyle\int_\Omega \tilde{T}_\sigma^+\Big(b_{S_n}(x,u_0^1) - b_{S_n}(x,u_0^2)\Big)\,dx\end{aligned}$$

where $\tilde{T}_\sigma^+(t) = \displaystyle\int_0^t T_\sigma^+(s)\,ds.$

Due to the assumption $u_0^1 \le u_0^2$ a.e. in Ω and the monotone character of $b_{S_n}(x,\ .)$ and $T_\sigma(.)$, we have

$$\int_\Omega \tilde{T}_\sigma^+\Big(b_{S_n}(x,u_0^1) - b_{S_n}(x,u_0^2)\Big)\,dx = 0 \tag{82}$$

It follows from (76), (81) and (82) that

$$\frac{1}{\sigma}\int_Q \tilde{T}_\sigma^+\Big(b_{S_n}(x,u_1) - b_{S_n}(x,u_2)\Big)\,dx\,dt + A_n^\sigma = B_n^\sigma + C_n^\sigma + D_n^\sigma \tag{83}$$

for any $\sigma > 0$ and any $n > 0$.

$\star$ *Step 2*. In this step, we study the behaviors of the terms A_n^σ, B_n^σ, C_n^σ and D_n^σ when σ tends to 0 and $n \to +\infty$. More precisely, we prove the following Lemma

Lemma 4.1. *We have*

$$\lim_{n\to+\infty}\ \underline{\lim}_{\sigma\to 0} A_n^\sigma \ge 0, \tag{84}$$

$$\underline{\lim}_{n\to+\infty}\ \underline{\lim}_{\sigma\to 0} B_n^\sigma = 0, \tag{85}$$

$$\underline{\lim}_{\sigma\to 0} C_n^\sigma = 0 \quad \textit{for all } n, \tag{86}$$

$$\underline{\lim}_{n\to+\infty}\ \overline{\lim}_{\sigma\to 0} D_n^\sigma \le 0. \tag{87}$$

Proof of Lemma 4.1. The lemma is proved in .[23]

In view of estimates (82), (83), (84), (85), (86) and (87) we have

$$\int_Q \Big(b(x,u_1) - b(x,u_2)\Big)^+ \, dx\, dt \le 0,$$

so that $b(x,u_1) \le b(x,u_2)$ a.e. in Q which in turn implies that $u_1 \le u_2$ a.e. in Q, theorem 4.1 will be then established.

References

1. P. Bénilan, L. Boccardo, T. Gallouët, R. Gariepy, M. Pierre and J.-L. Vazquez, *An L^1-theory of existence and uniqueness of solutions of nonlinear elliptic equations*, Ann. Scuola Norm. Sup. Pisa, **22**, (1995), 241-273.
2. D. Blanchard, *Truncation and monotonicity methods for parabolic equations equations,* Nonlinear Anal., **21**, (1993), 725-743.
3. D. Blanchard, and F. Murat, *Renormalized solutions of nonlinear parabolic problems with L^1 data, Existence and uniqueness*, Proc. Roy. Soc. Edinburgh Sect., **A 127**, (1997), 1137-1152.
4. D. Blanchard, F. Murat and H. Redwane, *Existence et unicité de la solution reormalisée d'un problème parabolique assez général*, C. R. Acad. Sci. Paris Sér., **I329**, (1999), 575-580.
5. D. Blanchard, F. Murat and H. Redwane, *Existence and Uniqueness of a Renormalized Solution for a Fairly General Class of Nonlinear Parabolic Problems*, J. Differential Equations, **177**, (2001), 331-374.
6. D. Blanchard and A. Porretta, *A Stefan problems with nonlinear diffusion and convection*, J. Diff. Equations, **210**, (2005), 383-428.
7. D. Blanchard and H. Redwane, *Renormalized solutions of nonlinear parabolic evolution problems*, J. Math. Pure Appl., **77**, (1998), 117-151.
8. L. Boccardo, D. Giachetti, J.-I. Diaz and F. Murat, *Existence and regularity of renormalized solutions for some elliptic problems involving derivation of nonlinear terms*, J. Differential Equations, **106**, (1993), 215-237.
9. L. Boccardo, F. Murat and J.-P. Puel, *Existence of bounded solutions for nonlinear elliptic unilateral problems*, Ann. Mat. Pura Appl., **152**, (1988), 183-196.
10. J. Carrillo, *Entropy solutions for nonlinear degenerate problems*, Arch. Ration. Mech. Anal., **147(4)**, (1999), 269-361.
11. J. Carrillo and P. Wittbold, *Uniqueness of renormalized solutions of degenerate elliptic-parabolic problems*, J. Differential Equations, **156**, (1999), 93-121.
12. J. Carrillo and P. Wittbold, *Renormalized entropy solution of a scalar conservation law with boundary condition*, J. Differential Equations, **185(1)**, (2002), 137-160.

13. A. Dall'Aglio and L. Orsina, *Nonlinear parabolic equations with natural growth conditions and L^1 data*, Nonlinear Anal., **27**, (1996), 59-73.
14. R. Landes, *On the existence of weak solutions for quasilinear parabolic initial-boundary value problems*, Proc. Roy. Soc. Edinburgh Sect., **A89**, (1981), 217-237.
15. J.-L. Lions, *Quelques méthodes de résolution des problèmes aux limites non linéaire*, Dunod et Gauthier-Villars, Paris, (1969).
16. R.-J.DiPerna and P.-L. Lions, *On the Cauchy problem for Boltzmann equations : Global existence and weak stability*, Ann. Math., **130**, (1989), 321-366.
17. J.-P. Lions, *Mathematical Topics in Fluid Mechanics, Vol. 1 : Incompressible models*, Oxford Univ. Press, (1996).
18. F. Murat, *Soluciones renormalizadas de EDP elipticas non lineales, Cours à l'Université de Séville*, Publication R93023, Laboratoire d'Analyse Numérique, Paris VI, (1993).
19. F. Murat, *Equations elliptiques non linéaires avec second membre L^1S ou mesure*, Comptes Rendus du 26ème Congrès National d'Analyse Numérique Les Karellis, (1994), A12-A24,
20. A. Porretta, *Existence results for nonlinear parabolic equations via strong convergence of trauncations*, Ann. Mat. Pura ed Applicata, **177**, (1999), 143-172.
21. N. Grenon, *Résultats d'existence et comportement asymptotique pour des équations paraboliques quasi-linéaire*, (1990). Thèse Université d'Orléans, France.
22. H. Redwane, *Existence of a solution for a class of parabolic equations with three unbounded nonlinearities*, Adv. Dyn. Syst. Appl., **2**, (2007), pp. 241-264.
23. H. Redwane, *Uniqueness of renormalized solutions for a class of parabolic equations with unbounded nonlinearities*, Rendiconti di Matematica, **VII**, (2008), pp. 189-200.
24. J. Simon, *Compact sets in $L^p(0,T;B)$*, Ann. Mat. Pura Appl., **146**, (1987), 65-96.

Existence and uniqueness of solutions of some nonlinear equations in Orlicz spaces and weighted Sobolev spaces

L. Aharouch

Faculté polydisciplinaire Ouarzazate
BP 638 Ouarzazate, Morocco
E-mail: laharouch@gmail.com

A. Benkirane*, J. Bennouna†, and A. Touzani‡

Département de Mathématiques et Informatique
Faculté des Sciences Dhar-Mahraz
B.P 1796 Atlas Fès, Morocco
*E-mails: *abd.benkirane@gmail.com, †jbennouna@hotmail.com,*
‡atouzani@menara.ma

In this paper, we show, in the framework of Orlicz Sobolev spaces, the existence of renormalized and entropy solutions of some nonlinear equations and the existence and uniqueness of a unilateral problem. The existence of weak and renormalized solution of a nonlinear equation is also presented in the framework of weighted Sobolev spaces.

Keywords: Orlicz Sobolev spaces; Weighted Sobolev spaces; Renormalized solution; Entropy solution; Weak solution; Boundary value problems.

1. Definition and existence of renormalized solutions in Orlicz space

Let Ω be a bounded open subset of $I\!R^N$ with the segment property.

Let M be an N-function satisfying the Δ_2-condition and let P be an N-function such that $P << M$.

Let $A : D(A) \subset W_0^1 L_M(\Omega) \to W^{-1} L_{\overline{M}}(\Omega)$ be a mapping (not defined everywhere) given by $A(u) = -div\ \ a(x, u, \nabla u)$, where $a : \Omega \times I\!R \times I\!R^N \to I\!R^N$ is a caratheodory function satisfying for $a.e.\ x \in \Omega$ and all $t \in I\!R, \xi, \overline{\xi}$ with $\xi \neq \overline{\xi}$

$$(1-1) \qquad | a(x, t, \xi) | \leq d(x) + k_1 \overline{P}^{-1} M(k_2 \mid t \mid) + k_3 \overline{M}^{-1} M(k_4 \mid \xi \mid),$$

$$(1-2) \qquad [a(x, t, \xi) - a(x, t, \overline{\xi})]\, [\xi - \overline{\xi}] > 0,$$

$$(1-3) \qquad a(x,t,\xi)\,\xi \geq \alpha M(\frac{|\,\xi\,|}{\lambda}),$$

where $d(x) \in E_{\overline{M}}(\Omega), d \geq 0, \alpha, \lambda \in I\!R^*_+, k_1, k_2, k_3, k_4 \in I\!R_+$.

Consider the nonlinear elliptic problem

$$(1-4) \qquad -div\ \ a(x,u,\nabla u) - div\phi(u) + g(x,u) = f\ \ in\ \Omega,$$

$$(1-5) \qquad u = 0\ \ on\,\partial\Omega,$$

where

$$(1-6) \qquad f \in W^{-1}E_{\overline{M}}(\Omega),$$

and $\phi = (\phi_1, \ldots, \phi_N)$ satisfy

$$(1-7) \qquad \phi \in (C^0(I\!R))^N.$$

Let $g(x,t)$ be a caratheodory function such that for $a.e.x \in \Omega$ and all $t \in I\!R$

$$(1-8) \qquad g(x,t)\,t \geq 0,$$

$$(1-9) \qquad \sup_{|t|\leq n}\ |\ g(.,t)\ | = h_n(.) \in L^1(\Omega) \quad \forall n.$$

Note that no growth hypothesis is assumed on the function ϕ, which implies that for a solution $u \in W^1_0 L_M(\Omega)$ the term $div\phi(u)$ may be meaningless, even as a distribution. As in[6] we define the following notion of renormalized solution, which gives a meaning to a possible solution of (1-4)-(1-5).

The notion of renormalized solutions in the usual sens was introduced by R.J. Diperna and P.L. Lions[10] for the study of the Boltzmann equations. This notion was then adapted to the study of the problem (1-1)-(1-2) by L. Boccardo, D. Giachetti, J.I. Diaz and F. Murat[6] when the right hand side is in $W^{-1,p'}(\Omega)$, by J.M. Rakotoson[12] when the right hand side is in $L^1(\Omega)$, and finally by G. Dal Maso, F. Murat, L. Orsina and A. Prignet[9] for the case of right hand side is general measure data.

Definition 1.1. Assume that $(1-1)-(1-3), (1-6)-(1-9)$ hold true. A function u is a renormalized solution of the problem $(1-4)-(1-5)$ if

$$(1-10) \qquad u \in W^1_0 L_M(\Omega), g(x,u) \in L^1(\Omega), u\,g(x,u) \in L^1(\Omega)$$

$$(1-11) \quad \begin{cases} -div\ a(x,u,\nabla u)\,h(u) - div(\phi(u)h(u)) + \phi(u)h'(u)\nabla u \\ \qquad\qquad +g(x,u)h(u) = fh(u) \quad \text{in } \mathcal{D}'(\Omega), \quad \forall h \in C^1_c(I\!R). \end{cases}$$

Remark 1.1. Let us note that in $(1-11)$ every term is meaningful in the distributional sense (in contrast with $(1-4)$).

Lemma 1.1. *Let Ω be a bounded open subset of $I\!R^N$ with the segment property. If $u \in (W_0^1 L_M(\Omega))^N$ then*

$$\int_\Omega div\, u\, dx = 0.$$

Theorem 1.1. *Let M be an N-function satisfying the Δ_2-condition. Under assumptions $(1-1)-(1-3), (1-6)-(1-9)$, there exists a renormalized solution u (in the sense of Definition 1.1) of problem $(1-4)-(1-5)$.*

Proof of Theorem 1.1 see[2]

In the last theorem, we have supposed the N-function M satisfying the Δ_2-condition. In the next theorem we prove the same result without any restriction on the N-function M (i.e. without the Δ_2-condition).

Theorem 1.2. *Under assumptions $(1-1)-(1-3), (1-6)-(1-9)$, there exists a renormalized solution u (in the sense of Definition 1.1) of problem $(1-4)-(1-5)$.*

Proof of Theorem 1.2 see[1]

2. Definition and existence of entropy solutions in Orlicz space

Let Ω be a bounded open subset of $I\!R^N$ with the segment property. Let M, P be two N-functions such that $P << M$.
Let $A : \mathcal{D}(A) \subset W_0^1 L_M(\Omega) \to W^{-1} L_{\overline{M}}(\Omega)$ be a mapping (not defined everywhere) given by: $A(u) = -\text{div} a(x, u, \nabla u)$ where $a : \Omega \times I\!R \times I\!R^N \to I\!R^N$ is a caratheodory function satisfying for a.e. x $\in \Omega$ and all $\xi, \overline{\xi} \in I\!R^N$ with $\xi \neq \overline{\xi}$:

$$(2-1) \qquad | a(x,t,\xi) | \leq d(x) + k_1 \overline{P}^{-1} M(k_2 \mid t \mid) + k_3 \overline{M}^{-1} M(k_4 \mid \xi \mid)$$

$$(2-2) \qquad [a(x,t,\xi) - a(x,t,\overline{\xi})]\, [\xi - \overline{\xi}] > 0$$

$$(2-3) \qquad a(x,t,\xi)\, \xi \geq \alpha M(\frac{|\, \xi \,|}{\lambda})$$

where $d(x) \in E_{\overline{M}}(\Omega), d \geq 0, \alpha, \lambda \in I\!R^*_+, k_1, k_2, k_3, k_4 \in I\!R_+$. Consider the nonlinear elliptic problem

$$(2-4) \qquad -\text{div}a(x,u,\nabla u) = f - div\phi(u)$$

$$(2-5) \qquad u = 0 \;\; on\, \partial\Omega$$

where

$$(2-6) \qquad f \in L^1(\Omega)$$

and $\phi = (\phi_1, \ldots, \phi_N)$ satisfy

$$(2-7) \qquad \phi \in (C^0(I\!R))^N.$$

Note that no growth hypothesis is assumed on the function ϕ, which implies that the term $div\phi(u)$ may be meaningless, even as a distribution. The notion of entropy solution, used in,[7] allows us to give a meaning to a possible solution of (2-4)-(2-5).

We introduce the following notation, see,[13][11]

Definition 2.1. Let M be an N-function, we define the following set:

$$\mathcal{A}_M = \{Q : \quad Q \text{ is an N-function such that } \frac{Q''}{Q'} \leq \frac{M''}{M'} \text{ and } \int_0^{\cdot} QoH^{-1}\left(\frac{1}{r^{1-\frac{1}{N}}}\right)dr < \infty \text{ where } H(r) = \frac{M(r)}{r}\}$$

Remark 2.1. Let $M(t) = t^p$ and $Q(t) = t^q$, then the condition $Q \in \mathcal{A}_M$, is equivalent to the following conditions:
i) $2 - \frac{1}{N} < p < N$,
ii) $q < \tilde{p} = \frac{(p-1)N}{N-1}$

Definition 2.2. Assume that $(2-1)-(2-3), (2-6)-(2-7)$ hold true, and suppose that $\mathcal{A}_M \neq \emptyset$. A function u is an entropy solution of problem $(2-4)-(2-5)$ if

$$\begin{cases} u \in W_0^1 L_Q(\Omega) \qquad \forall Q \in \mathcal{A}_M, \\ T_k(u) \in W_0^1 L_M(\Omega) \qquad \forall k > 0, \\ \int_\Omega a(x,u,\nabla u)\, \nabla T_k[u-v]dx \leq \int_\Omega f T_k[u-v]dx + \int_\Omega \phi(u)\, \nabla T_k[u-v]dx \\ \forall v \in W_0^1 L_M(\Omega) \cap L^\infty(\Omega). \end{cases}$$

Theorem 2.1. *Assume that* $(2-1)-(2-3), (2-6)-(2-7)$ *hold true, and suppose that* $\mathcal{A}_M \neq \emptyset$*, there exists an entropy solution* u *of problem* $(2-4)-(2-5)$ *(in the sense of Definition 2.1).*

Proof of Theorem 2.1 see[3]

Remark 2.2. In the case $M(t) = t^p$, our theorem gives a refinement of the regularity result (i.e. $u \in W_0^{1,q}(\Omega), q < \tilde{p} = \frac{(p-1)N}{N-1}$).

In fact by Theorem 2.1 we have $u \in W_0^1 L_Q(\Omega)$ $\forall Q \in \mathcal{A}_M$ (for example for $Q(t) = \frac{t^{\tilde{p}}}{log^{\alpha}(e+t)},\ \alpha > 1$).

3. Existence and uniqueness of solution of unilateral problems with L^1-data in Orlicz spaces

Let Ω be a bounded open subset of $I\!R^N$ with the segment property.

Let M be an N-function, satisfying the Δ_2-condition.

Let $A : D(A) \subset W_0^1 L_M(\Omega) \to W^{-1} L_{\overline{M}}(\Omega)$ be a mapping (not defined everywhere) given by: $A(u) = -div\ \ a(x, \nabla u)$ where $a : \Omega \times I\!R^N \to I\!R^N$is a caratheodory function satisfying for a.e. x $\in \Omega$ and all $\xi, \overline{\xi} \in I\!R^N$ with $\xi \neq \overline{\xi}$:

$$(3-1) \qquad | a(x,\xi) | \leq d(x) + k_1 \overline{M}^{-1} M(k_2 | \xi |)$$

$$(3-2) \qquad [a(x,\xi) - a(x,\overline{\xi})]\,[\xi - \overline{\xi}] > 0$$

$$(3-3) \qquad a(x,\xi)\,\xi \geq \alpha M(| \xi |)$$

where $d(x) \in E_{\overline{M}}(\Omega), d \geq 0, \alpha, k_1, k_2 \in I\!R_+$.
Let

$$K_\psi = \{v \in W_0^1 L_M(\Omega) \cap L^\infty(\Omega) : v \geq \psi \text{ a.e. in } \Omega\}$$

where $\psi : \Omega \to I\!R_+$ is a measurable function on Ω such that:

$$\psi \in W_0^1 L_M(\Omega) \cap L^\infty(\Omega).$$

Assume that

$$(3-4) \qquad f \in L^1(\Omega).$$

We suppose the regularity assumption on the obstacle function ψ:

$$(3-5) \qquad \text{there exists } \overline{\psi} \in K \text{ such that } \psi - \overline{\psi} \text{ is continuous on } \Omega.$$

Consider the nonlinear elliptic unilateral problem:

$$(3-6) \quad \begin{cases} u \in W_0^1 L_Q(\Omega) & \forall Q \in \mathcal{A}_M \\ u \geq \psi \quad \text{a.e. in } \Omega & \\ T_k(u) \in W_0^1 L_M(\Omega) & \forall k > 0 \\ \int_\Omega a(x,\nabla u)\, \nabla T_k[u-v]dx \leq \int_\Omega f T_k[u-v]dx & \forall v \in K_\psi. \end{cases}$$

It is our purpose, in this paper, to show the existence and uniqueness of solutions for the problem (3-6) in the setting of the Orlicz Sobolev space $W_0^1 L_M(\Omega)$. Our result, theorem 3.1, generalizes that of Boccardo[8] and gives in particular a refinement of his result.

Theorem 3.1. *Assume that* $(3-1)-(3-5)$ *hold true, and suppose that* $\mathcal{A}_M \neq \emptyset$*, then there exists an unique solution of problem* $(3-6)$*.*

Proof of Theorem 3.1 see[4]

4. Existence of solutions for nonlinear elliptic degenerated equations

Let Ω be a bounded open subset of $I\!R^N$. A is a nonlinear operator of the Leray-lions type from a weighted Sobolev space $W^{1,p}(\Omega,\nu)$ (where $\nu=\nu(x)$ is weight function defined on Ω). A is defined by $A(u)=-div\ \ a(x,u,\nabla u)$ where a is caratheodory function from $\Omega\times I\!R\times I\!R^N\to I\!R^N$ satisfying for a.e. $x\in\Omega$, and for all ξ_0,ξ,

$$(4-1)\qquad \mid a(x,\xi_0,\xi)\mid\leq K(\mid\xi_0\mid)\nu^{\frac{1}{p}}(x)\{d(x)+\mid\xi_0\mid^{p-1}+\nu^{1-\frac{1}{p}}\mid\xi\mid^{p-1}\},$$

where K verify that $\underset{|s|\leq k}{sup\,ess}\,K(\mid s\mid)<\infty,\qquad \forall k>0$, and $d(x)\in L^{p'}(\Omega)$.

$$(4-2)\qquad [a(x,\xi_0,\xi)-a(x,\xi_0,\eta)]\,[\xi-\eta]>0\ \text{ where } \xi\neq\eta.$$

Let λ be a continuous function such that $\lambda>0$ and defined on $I\!R_+$, the degeneracy of the operator A is expressed by the assumption

$$(4-3)\qquad a(x,\xi_0,\xi)\,\xi\geq\nu(x)\,\lambda(\mid\xi_0\mid)\ \mid\xi\mid^p,$$

holds for all ξ_0,ξ and let λ_1 such that

$$(4-4)\qquad \lambda_1(s)=\int_0^s[\lambda(t)]^{\frac{p'}{p}}dt,\quad s\geq 0\quad \text{with } \lambda_1(\infty)=\infty.$$

Furthermore, we shall assume that for some $m_0>0$,

$$(4-5)\qquad \inf_{m\geq m_0}(m^{p.(1-\frac{p'}{r})}[\inf_{m\leq t\leq m+1}\lambda(t)])>0.$$

For instance $\lambda(t)=t^{\frac{-\gamma p}{p'}}$ for $t\geq 1$, $0<\gamma<1$ (small).
We will suppose that

$$(4-6)\qquad \nu\in L^1_{loc}(\Omega),\ \ \nu^{\frac{-1}{p-1}}\in L^1(\Omega),$$

$$(4-7) \qquad \nu^{-s} \in L^1(\Omega) \quad \text{with } \ s \in (\frac{N}{p}, \infty) \cap [\frac{1}{p-1}, \infty).$$

We consider the nonlinear elliptic problem

$$(4-8) \qquad -div \ \ a(x,u,\nabla u) - div\phi(u) + g(x,u) = f,$$

$$(4-9) \qquad u = 0 \ \ on\, \partial\Omega,$$

where $\phi = (\phi_1, ..., \phi_N)$ satisfy

$$(4-10) \qquad \phi \in (C^0(I\!R))^N.$$

Note that no growth hypothesis is assumed on the function ϕ. Let $g(x,t)$ be a caratheodory function such that for $a.e.x \in \Omega$ and all $t \in I\!R$,

$$(4-11) \qquad g(x,t)\, t \geq 0$$

$$(4-12) \qquad \sup_{|t| \leq n} \ | \ g(.,t) \ |= h_n(.) \in L^1(\Omega) \ \forall n.$$

The right-hand side in (4-8) is of the form

$$(4-13) \qquad f = \sum_{i=1}^{N} \frac{\partial f_i}{\partial x_i},$$

where the family $\{f_i \ \ i = 1, ..., N\}$ satisfies the conditions

$$f_i \in L^r(\Omega, \nu^{\frac{-r}{p}}) \hookrightarrow L^{p'}(\Omega, \nu^{\frac{-p'}{p}}) = (L^p(\Omega, \nu))^* \text{ for } i = 1, \ldots, N, \text{ with } r \geq p'.$$

We distinguish two cases.

The first case if $r > r_c$ with $r_c = \frac{q}{q-N} . \frac{N}{p-1}$ and $q > N$. In this case we prove in Theorem 4.1 that the solution u of problem (4-8)-(4-9) is bounded, and then the term $div\phi(u)$ has meaningful as a distribution. In Theorem 4.2, we prove that the last problem admits a weak solution (see Definition 4.1).

The second case if $p' \leq r \leq r_c$ the solution u is not bounded and the term $div\phi(u)$ may be meaningless, even as a distribution. As in[6] we define the notion of renormalized solution (see Definition 4.2), which gives a meaning to a possible solution of (4-8)-(4-9). In Theorem 4.3 we prove the existence of renormalized solution.

We show in the paper L^∞ estimates for the solutions (see Theorem 4.1) and the existence of a weak and renormalized solution, and is also showed the fact that the main operator is degenerate in the space variable.

4.1. *Regularity and existence of weak solutions*

Definition 4.1. Assume that $(4-1)-(4-7),(4-10)-(4-13)$ hold true.A function u is a weak solution of the problem $(4-8)-(4-9)$ if

$$(4-14) \qquad u \in W_0^{1,p}(\Omega,\nu)\cap L^\infty(\Omega), g(x,u) \in L^1(\Omega), u\, g(x,u) \in L^1(\Omega)$$

$$(4-15) \int_\Omega a(x,u,\nabla u)\nabla\varphi dx + \int_\Omega \phi(u)\nabla\varphi dx + \int_\Omega g(x,u)\varphi dx = \sum_{i=1}^N \int_\Omega \frac{\partial f_i}{\partial x_i}\varphi dx$$

$\forall \varphi \in \mathcal{D}(\Omega)$.

In Theorem 4.1 we prove the boundness of the solution of the problem (4-8)-(4-9), in Theorem 4.2 we prove the existence of weak solution if $r > r_c$.

Lemma 4.1. *Let Ω be a bounded open subset of $I\!R^N$. If $u \in (W_0^{1,p}(\Omega,\nu))^N$, then*

$$\int_\Omega div\, u\, dx = 0.$$

Lemma 4.2. *Let $F : I\!R \to I\!R$ be uniformly lipschitzian with $F(0) = 0$. Let $u \in W_0^{1,p}(\Omega,\nu)$, then $F(u) \in W_0^{1,p}(\Omega,\nu)$.*

Let $u \in W_0^{1,p}(\Omega,\nu)$ and let $S_{\theta,h}$ a real lipschitzian function defined for $\theta > 0, h > 0$ by

$$S_{\theta,h}(\tau) = \begin{cases} 1 & \text{if } \tau \geq \theta + h \\ \frac{\tau-\theta}{h} & \text{if } \theta \leq \tau \leq \theta + h \\ 0 & \text{if } \mid \tau \mid \leq \theta \\ \frac{\tau+\theta}{h} & \text{if } -\theta - h \leq \tau \leq -\theta \\ -1 & \text{if } \tau \leq -\theta - h \end{cases}$$

then $S_{\theta,h}(u) \in W_0^{1,p}(\Omega,\nu)$ by Lemma 4.2 and we suppose that u satisfies $\forall\theta \in]0, \sup ess \mid u \mid [, \forall h \in]0, \sup ess \mid u \mid -\theta[$,

$$(4-16) \quad \begin{aligned} &\int_\Omega a(x,u,\nabla u)\nabla S_{\theta,h}(u)dx + \int_\Omega \phi(u)\nabla S_{\theta,h}(u) + \int_\Omega g(x,u)S_{\theta,h}(u)dx \\ &= \sum_{i=1}^N \int_\Omega \frac{\partial f_i}{\partial x_i} S_{\theta,h}(u)dx \end{aligned}$$

Remark 4.1. It is easy to see that each possible solution of problem (4-8)-(4-9) verifies the relation (4-16).

Theorem 4.1. *Under assumptions* $(4-1)-(4-7),(4-10)-(4-13)$. *Let* $u \in W_0^{1,p}(\Omega,\nu)$ *which satisfies (4-16), and we suppose furthermore* $f_i \in L^r(\Omega, \nu^{\frac{-r}{p}})$, *with* $r > r_c$ *where* $r_c = \frac{q}{q-N} \cdot \frac{N}{p-1}$ *and* $q > N$. *Then* u *is bounded and we have the following estimate*

$$(4-17) \quad \lambda_1(\| u \|_\infty) \leq C_N \alpha_1 \alpha_2 \alpha_3 \;\Leftrightarrow \| u \|_\infty \leq \lambda_1^{-1}(C_N \alpha_1 \alpha_2 \alpha_3) = M,$$

where

$$\begin{aligned} &\alpha_1 = \| \nu^{\frac{-1}{p}} \|_{L^q(\Omega)}, \\ &\alpha_2 = \Big(\sum_{i=1}^N \| f_i \|_{L^r(\Omega,\nu^{\frac{-r}{p}})}\Big)^{\frac{1}{p-1}}, \\ &\alpha_3 = \Big(\int_0^{|\Omega|} \sigma^{m(1-\frac{1}{N})} d\sigma\Big)^{\frac{1}{m}}, \quad \text{with } \frac{1}{m} + \frac{1}{r(p-1)} + \frac{1}{q} = 1, \end{aligned}$$

$C_N = \frac{1}{N \alpha_N^{\frac{1}{N}}}$, α_N *is the measure of the unit ball of* $\mathbb{R}^N$,
λ_1^{-1} *is the converse of the function* λ_1 *(see (4-4)).*

Proof of Theorem 4.1. The proof is based on the method of relative rearrangement see[5] .

Theorem 4.2. *Under assumptions* $(4-1)-(4-7),(4-10)-(4-13)$. *There exist a bounded weak solution* $u \in W_0^{1,p}(\Omega,\nu) \cap L^\infty(\Omega)$ *(in the sense of Definition* 4.1*) of problem* $(4-8)-(4-9)$.

Proof of Theorem 4.2. See[5] .

4.2. *Existence of renormalized solutions*

If $p' \leq r \leq r_c$, the solution of the problem (4-8)-(4-9) may not be bounded and the term $div(\phi(u))$ may have no meaning even a distribution. Then we consider the nonlinear elliptic problem (4-8)-(4-9) in Ω, and we consider the following notion of renormalized solution which gives a meaning to a possible solution of (4-8)-(4-9).

Definition 4.2. Assume that $(4-1)-(4-7),(4-10)-(4-13)$ hold true. A function u is a renormalized solution of the problem $(4-8)-(4-9)$ if

$$u \in W_0^{1,p}(\Omega,\nu), g(x,u) \in L^1(\Omega), u\, g(x,u) \in L^1(\Omega)$$

$$\left\{\begin{array}{l} \displaystyle\int_{\Omega} a(x,u,\nabla u)\,[h^{'}(u)\nabla u\varphi + h(u)\nabla\varphi]dx + \int_{\Omega}\Phi(u)[h^{'}(u)\nabla u\varphi \\ \displaystyle + h(u)\nabla\varphi]dx + \int_{\Omega} g(x,u)h(u)\varphi dx = \sum_{i=1}^{N}\int_{\Omega}\frac{\partial f_i}{\partial x_i}h(u)\varphi dx \\ \forall h \in C_c^1(I\!R), \varphi \in W_0^{1,p}(\Omega,\nu)\cap L^{\infty}(\Omega) \end{array}\right.$$

In Theorem 4.3 we state the existence of renormalized solution of (4-8)-(4-9).

Theorem 4.3. *Under assumptions* $(4-1)-(4-7), (4-10)-(4-13)$*, and furthermore for* $\frac{N}{p-1} < r \leq r_c$ *with* $q > N$*, there exists a renormalized solution* u *(in the sense of Definition* 4.2*) of problem* $(4-8)-(4-9)$

Proof of Theorem 4.3 is given in[5] .

References

1. L. Aharouch, J. Bennouna and A. Touzani, *Existence of renormalized solution of some elliptic problems in Orlicz spaces,* Revista Matematica Complutense, 22 (2009), n1, 91-110.
2. A. Benkirane, J. Bennouna, *Existence of Renormalized Solutions for some elliptic problems involving derivatives of nonlinear terms in Orlicz Spaces,* Lectures notes in pure and applied mathematics , 229, Dekker, New York, 2002.
3. A. Benkirane, J. Bennouna, *Existence of Entropy Solutions for some nonlinear problems in* L^1 *in Orlicz Spaces,* Abstract and Applied Analysis 7 (2002) n 2, 85-102 USA.
4. A. Benkirane, J. Bennouna, *Existence and uniqueness of solution of unilateral problems with* L^1*-data in Orlicz spaces,* Ital. J. of Pure and App. Math., **16**,(2004) 87-102.
5. A. Benkirane, J. Bennouna, *Existence of Solutions for nonlinear elliptic degenerate equations, Nonlinear Analysis 54 (2003), n9-37, USA.*
6. L. Boccardo, D. Giachetti, J.I. Diaz, F. Murat, *Existence and Regularity of Renormalized Solutions of some Elliptic Problems involving derivatives of nonlinear terms, Journal of differential equations 106,215-237 (1993)*
7. L. Boccardo, *Some nonlinear Dirichlet problems in* L^1 *involving lower order terms in divergence form, progress in Elliptic and Parabolic Partiel Differentiel Equations (capri, 1994), Pitman Res. Notes Math. Ser.,* **vol. 350**, Longman, Harlow, 1996, pp. 43-57.
8. L. Boccardo and G. R. Cirmi, *Existence and uniqueness of solution of unilateral problems with* L^1 *data,* Journal of Convex Analysis, **6**, N1 (1999) 195-206.
9. G. Dalmaso, F. Murat, L. Orsina and A. Prignet, *Renormalized solutions of elliptic equations with general measure data,* Ann. Scuola Norm. Sup Pisa Cl. Sci 12 4 (1999) 741-808.

10. R.J. DIPERNA, P.L. LIONS, *on the cauchy problem for Boltzmann equations: Global existence and weak stability, ann of Math, 130 (1989), 321-366.*
11. M. KBIRI ALAOUI, *Sur certains problems elliptiques avec second membre mesure ou* L^1_{loc}, These soutenu à l'Université Sidi Moh. Ben Abd. Faculté des Sciences Dhar Mehraz Fes Maroc (1999)
12. J.M. Rakotoson, *Uniqueness of renormalized solutions in a* T*-set for* L^1 *data problems and the link between various formulations*, Indiana University Math. Jour., vol. 43, 2(1994).
13. G. TALENTI, *Nonlinear Elliptic Equations, Rearrangements of functions and Orlicz spaces*, Ann. Mat. Pura Appl.(4) 120 (1979), 159-184.

Existence of solutions for variational degenerated unilateral problems

L. Aharouch, E. Azroul, and M. Rhoudaf

Département de Mathématiques et Informatique
Faculté des Sciences Dhar-Mahraz
B.P 1796 Atlas Fès, Morocco

An existence result is proved for a variational degenerated unilateral problems associated to the following equations

$$Au + g(x, u, \nabla u) = f,$$

where A is a Leray-Lions operator acting from the weighted Sobolev space $W_0^{1,p}(\Omega, w)$ into its dual $W^{-1,p'}(\Omega, w^*)$, while $g(x, s, \xi)$ is a nonlinear term which has a growth condition with respect to ξ and a sign condition on s, i.e. $g(x, s, \xi).s \geq 0$ for every $s \in \mathbb{R}$ and for every x and ξ in their respective domains. The source term f is supposed to belong to $W^{-1,p'}(\Omega, w^*)$.

Keywords: Degenerate unilateral problem; Existence result.

1. Introduction

Let Ω be a bounded open subset of $\mathbb{R}^N (N \geq 2)$, p be a real number such that $1 < p < \infty$ and $w = \{w_i(x); 0 \leq i \leq N\}$, be a collections of weight functions on Ω, i.e. each $w_i(x)$ is a measurable a.e. strictly positive function on Ω satisfying some intergrability conditions (see section 2).

In this paper we are interested in the study of the degenerated obstacle problem associated to the following Dirichlet problem

$$\begin{cases} Au + g(x, u, \nabla u) = f \quad \text{in} \quad \Omega \\ u \equiv 0 \quad \text{on} \quad \partial\Omega, \end{cases} \tag{1}$$

where $Au = -\text{div}(a(x, u, \nabla u))$ is a Leray-Lions operator acting from $W_0^{1,p}(\Omega, w)$ into its dual $W^{-1,p'}(\Omega, w^*)$ with $w^* = \left\{w_i^{1-p'}; 0 \leq i \leq N\right\}$, $p' = \frac{p}{p-1}$ is the conjugate exponent of p and where $g(x, u, \nabla u)$ is a nonlinearity term satisfying some p-growth condition with respect to ∇u, and satisfies the sign condition $g(x, u, \nabla u)u \geq 0$, but has unrestricted growth with respect to u.

The source term f is assumed to belong to $W^{-1,p'}(\Omega, w^*)$.

Let us start by the case of equation and recall that in the particular case where $g(x, u, \nabla u) = -C_0|u|^{p-2}u$ the following degenerated equation

$$-\text{div}(a(x, u, \nabla u)) - C_0|u|^{p-2}u = h(x, u, \nabla u),$$

has been studied by Drabek-Nicolosi[9] under some more degeneracy and some additional assumptions on h and a. While the existence solution for the variational Dirichlet problem (1) is treated in the work[3] but under the following integrability condition

$$\sigma^{-1/q-1} \in L^1_{loc}(\Omega) \quad \text{for some} \quad q \quad \text{such that} \quad 1 < q < \infty, \tag{2}$$

where q is the so-called Hardy exponent and σ is the so-called Hardy weight (see (14)) below).

Now we turn our attention to the degenerated unilateral case and we will give some known about the following problem

$$\begin{cases} \text{Find} \quad u \in K_\psi, g(x, u, \nabla u) \in L^1(\Omega), g(x, u, \nabla u)u \in L^1(\Omega) \\ \displaystyle\int_\Omega a(x, u, \nabla u)\nabla(u - v)\, dx + \int_\Omega g(x, u, \nabla u)(u - v)\, dx \\ \qquad\qquad\qquad\qquad \leq \langle f, u - v\rangle, \\ \forall \quad v \in K_\psi \cap L^\infty(\Omega), \end{cases} \tag{3}$$

where the convex set K_ψ is defined as

$$K_\psi = \{v \in W_0^{1,p}(\Omega, w);\ v \geq \psi \quad a.e. \text{ in } \ \Omega\},$$

with an obstacle ψ which is a measurable function on Ω.

Akdim et al. have proved in[4] the existence of a solution for the problem (3) under the following conditions

$$\sigma^{-\frac{1}{q-1}} \in L^1_{loc}(\Omega) \quad \text{with} \quad 1 < q < p + p', \tag{4}$$

and

$$\psi^+ \in L^\infty(\Omega) \cap W_0^{1,p}(\Omega, w). \tag{5}$$

For that, the authors in[4] have approximated the nonlinear term g by some function involving $\chi_{\Omega_\varepsilon}$ where Ω_ε is a sequence of compacts covering the bounded open set Ω and $\chi_{\Omega_\varepsilon}$ is a characteristic function, i.e.

$$g_\varepsilon(x, s, \xi) = \frac{g(x, s, \xi)}{1 + \varepsilon|g(x, s, \xi)|}\chi_{\Omega_\varepsilon}(x).$$

So there are consider the following approximate unilateral problem

$$\begin{cases} u_\varepsilon \in K_\psi \quad \text{such that} \\ \displaystyle\int_\Omega a(x, u_\varepsilon, \nabla u_\varepsilon)\nabla(u_\varepsilon - v)\, dx + \int_\Omega g_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon)(u_\varepsilon - v)\, dx \\ \qquad\qquad\qquad\qquad\qquad\qquad \leq \langle f, u_\varepsilon - v\rangle, \\ \forall \ \ v \in K_\psi, \end{cases} \tag{6}$$

Note that, the hypotheses (4) and (5) used in[4] have played an important role for to assure the boundedness, coercivity and pseudo-monotonicity of the operator associated to the approximate problem (6) and also for to prove the boundedness of the approximate solution u_ε in the space $W_0^{1,p}(\Omega, w)$ (see[3] for more details).

The aim of this paper is then to study the existence solution for the same degenerated unilateral problem (3) but without assuming the condition (4) nor (5).

To overcome the difficulties mentioned above, we have changed in the present paper the classical coercivity,

$$a(x, s, \xi)\xi \geq \alpha \sum_{i=1}^{N} w_i(x)|\xi_i|^p$$

by the following one,

$$a(x, s, \xi)(\xi - \nabla v_0) \geq \alpha \sum_{i=1}^{N} w_i(x)|\xi_i|^p - \delta(x), \tag{7}$$

where v_0 is some element of $K_\psi \cap L^\infty(\Omega)$ and $\delta(x)$ is an element of $L^1(\Omega)$ and also we have replaced the approximation term g_ε (of the nonlinearity g) by the following one,

$$g_n(x, s, \xi) = \frac{g(x, s, \xi)}{1 + \frac{1}{n}|g(x, s, \xi)|}\theta_n(x), \tag{8}$$

where the function $\theta_n(x)$ is defined according to the Hardy weight σ and hardy exponent q, i.e.,

$$\theta_n(x) = nT_{1/n}(\sigma^{1/q}(x)).$$

($T_{1/n}$ is the truncation operator at height $1/n$ see (18)).

It would be interesting at this stage to refer the reader to our previous work[1] in which the same unilateral problem with L^1-data is studied under the integrability condition (4) and under the regularity condition (5). Another work in this direction can be found in[2] where the existence solution of the same unilateral problem (with L^1-data) is proved by assuming the previous hypotheses (4) and (5), but the sign condition (22) is violated and the classical growth condition of the nonlinearity g (see (23)) is replaced by

$$|g(x,s,\xi)| \le c(x) + \rho(s)\sum_{i=1}^{N} w_i|\xi_i|^p, \tag{9}$$

(with $\rho \in L^1(\Omega), \rho \ge 0$).

We refer also to the work,[14] where the authors have solved an analogous problem in the classical Sobolev space but where the obstacle function is supposed to satisfy the condition $\psi^+ \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$.

The outline of this paper is as follows. After giving some preliminary results about the weighted Sobolev space in section 2, we formulate in section 3 our problem and we give the main existence result, which its proof is giving in section 4. And we achieve the paper by an appendix in section 5 where some intermediate results are proved.

2. Preliminaries

Let Ω be a bounded open subset of $I\!R^N (N \ge 2)$. Let $1 < p < \infty$ and let $w = \{w_i(x); i = 0, \ldots, N\}$, be a vector of weight functions, i.e., every component $w_i(x)$ is a measurable function which is strictly positive a.e. in Ω. Further, we suppose in all our considerations that,

$$w_i \in L^1_{loc}(\Omega) \tag{10}$$

and

$$w_i^{-\frac{1}{p-1}} \in L^1_{loc}(\Omega), \quad \text{for} \quad 0 \le i \le N. \tag{11}$$

We define the weighted space with weight γ in Ω as,

$$L^p(\Omega,\gamma) = \{u(x), u\gamma^{\frac{1}{p}} \in L^p(\Omega)\},$$

which is normed by,

$$\|u\|_{p,\gamma} = \left(\int_\Omega |u(x)|^p \gamma(x)\, dx\right)^{\frac{1}{p}}.$$

We denote by $W^{1,p}(\Omega, w)$ the weighted Sobolev space of all real-valued functions $u \in L^p(\Omega, w_0)$ such that the derivatives in the sense of distributions satisfy,

$$\frac{\partial u}{\partial x_i} \in L^p(\Omega, w_i) \quad \text{for all} \quad i = 1, \ldots, N.$$

This set of functions forms a Banach space under the norm

$$\|u\|_{1,p,w} = \left(\int_\Omega |u(x)|^p w_0 \, dx + \sum_{i=1}^N \int_\Omega |\frac{\partial u}{\partial x_i}|^p w_i(x) \, dx \right)^{\frac{1}{p}}. \tag{12}$$

To deal with the Dirichlet problem, we use the space $W_0^{1,p}(\Omega, w)$ defined as the closure of $C_0^\infty(\Omega)$ with respect to the norm (12). Note that $C_0^\infty(\Omega)$ is dense in $W_0^{1,p}(\Omega, w)$ and $(W_0^{1,p}(\Omega, w), \|.\|_{1,p,w})$ is a reflexive Banach space.

We recall that the dual of the weighted Sobolev spaces $W_0^{1,p}(\Omega, w)$ is equivalent to $W^{-1,p'}(\Omega, w^*)$, where $w^* = \{w_i^* = w_i^{1-p'}\}, \;\; i = 1, \ldots, N$ and p' is the conjugate of p, i.e., $p' = \frac{p}{p-1}$. For more details we refer the reader to.[11]

Now, we state the following assumptions.

(H_1) The expression,

$$\|u\| = \left(\sum_{i=1}^N \int_\Omega |\frac{\partial u}{\partial x_i}|^p w_i(x) \, dx \right)^{\frac{1}{p}} \tag{13}$$

is a norm on $W_0^{1,p}(\Omega, w)$ equivalent to the norm (12).
There exists a weight function σ strictly positive a.e. in Ω and a parameter $q, \; 1 < q < \infty$, such that the Hardy inequality

$$\left(\int_\Omega |u|^q \sigma(x) \, dx \right)^{\frac{1}{q}} \le C \left(\sum_{i=1}^N \int_\Omega |\frac{\partial u}{\partial x_i}|^p w_i(x) \, dx \right)^{\frac{1}{p}}, \tag{14}$$

holds for every $u \in W_0^{1,p}(\Omega, w)$ with a constant $C > 0$ independent of u. Moreover, the imbedding

$$W_0^{1,p}(\Omega, w) \hookrightarrow L^q(\Omega, \sigma) \tag{15}$$

determined by the inequality (14) is compact.

Note that, $(W_0^{1,p}(\Omega, w), \| \; \|)$ is a uniformly convex (and thus reflexive) Banach space.

Remark 2.1. Assume that $w_0(x) = 1$ and in addition the integrability condition:
There exists $\nu \in]\frac{N}{p}, \infty[\cap [\frac{1}{p-1}, \infty[$ such that $w_i^{-\nu} \in L^1(\Omega)$ holds for all $i = 1, \ldots, N$ (which is stronger than (11)). Then

$$\|u\| = \left(\sum_{i=1}^N \int_\Omega |\frac{\partial u}{\partial x_i}|^p w_i(x) \, dx \right)^{\frac{1}{p}}$$

is a norm defined on $W_0^{1,p}(\Omega, w)$ and it is equivalent to (12). Moreover

$$W_0^{1,p}(\Omega, w) \hookrightarrow\hookrightarrow L^q(\Omega)$$

for all $1 \leq q < p_1^*$ if $p\nu < N(\nu+1)$ and for all $q \geq 1$ if $p\nu \geq N(\nu+1)$, where $p_1 = \frac{p\nu}{\nu+1}$ and $p_1^* = \frac{Np_1}{N-p_1} = \frac{Np\nu}{N(\nu+1)-p\nu}$ is the Sobolev conjugate of p_1 (see[11]). Thus the hypotheses (H_1) is satisfied for $\sigma \equiv 1$.

Remark 2.2. If we use the special weight functions w and σ expressed in terms of the distance to the boundary $\partial\Omega$. Denote $d(x) = \mathrm{dist}(x, \partial\Omega)$ and set

$$w(x) = d^\lambda(x), \quad \sigma(x) = d^\mu(x).$$

In this case, the Hardy inequality reads

$$\left(\int_\Omega |u|^q d^\mu(x)\, dx\right)^{\frac{1}{q}} \leq C \left(\int_\Omega |\nabla u|^p d^\lambda(x)\, dx\right)^{\frac{1}{p}}.$$

(i) For, $1 < p \leq q < \infty$,

$$\lambda < p-1, \quad \frac{N}{q} - \frac{N}{p} + 1 \geq 0, \quad \frac{\mu}{q} - \frac{\lambda}{p} + \frac{N}{q} - \frac{N}{p} + 1 > 0. \tag{16}$$

(ii) For, $1 \leq q < p < \infty$,

$$\lambda < p-1, \quad \frac{\mu}{q} - \frac{\lambda}{p} + \frac{1}{q} - \frac{1}{p} + 1 > 0. \tag{17}$$

The conditions (16) or (17) are sufficient for the compact imbedding (15) to hold (see for example [,[10] Example 1], [,[11] Example 1.5, p.34], and [,[16] theorems 19.17 and 19.22]).

Now, we give the following technical lemmas which are needed later.

Lemma 2.1. *cf.*[3,15] *Let $g \in L^r(\Omega, \gamma)$ and let $g_n \in L^r(\Omega, \gamma)$, with $\|g_n\|_{\Omega,\gamma} \leq c$, $1 < r < \infty$. If $g_n(x) \to g(x)$ a.e. in Ω, then $g_n \rightharpoonup g$ weakly in $L^r(\Omega, \gamma)$.*

Lemma 2.2. *cf.*[3,12] *Assume that (H_1) holds. Let $F : I\!R \to I\!R$ be uniformly Lipschitzian, with $F(0) = 0$. Let $u \in W_0^{1,p}(\Omega, w)$. Then $F(u) \in W_0^{1,p}(\Omega, w)$. Moreover, if the set D of discontinuity points of F' is finite, then*

$$\frac{\partial(F \circ u)}{\partial x_i} = \begin{cases} F'(u)\frac{\partial u}{\partial x_i} & a.e. \quad in \quad \{x \in \Omega : u(x) \notin D\} \\ 0 & a.e. \quad in \quad \{x \in \Omega : u(x) \in D\}. \end{cases}$$

We introduce the truncation operator. For a given constant $k > 0$ we define the cut function $T_k : I\!R \to I\!R$ as

$$T_k(s) = \begin{cases} s & \text{if } \ |s| \leq k \\ k \ \text{sign}(s) & \text{if } \ |s| > k. \end{cases} \tag{18}$$

For a function $u = u(x), x \in \Omega$, we define the truncated function $T_k u = T_k(u)$ pointwise: for every $x \in \Omega$ the value of $(T_k u)$ at x is just $T_k(u(x))$. From Lemma 2.2, we deduce the following.

Lemma 2.3. *cf.*[3] *Assume that* (H_1) *holds. Let* $u \in W_0^{1,p}(\Omega, w)$ *and let* $T_k(u)$ *be the usual truncation* ($k \in I\!R^+$)*. Then* $T_k(u) \in W_0^{1,p}(\Omega, w)$*. Moreover, we have*

$$T_k(u) \to u \ \ \textit{strongly in} \ \ W_0^{1,p}(\Omega, w).$$

We states that every element of $W^{-1,p'}(\Omega, w^*)$ can be decomposed as $f_0 - \text{div}F$ where $f_0 \in L^{p'}(\Omega, w_0^{1-p'})$ and $F \in \prod_{i=1}^{N} L^{p'}(\Omega, w_i^{1-p'})$.

3. Main results

Let Ω be an open bounded subset of $I\!R^N$, $N \geq 2$. Given a measurable function $\psi : \Omega \to \overline{I\!R}$ (so-called obstacle) and consider its associated convex set,

$$K_\psi = \{u \in W_0^{1,p}(\Omega, w); \ u \geq \psi \ \ a.e. \text{ in } \ \Omega\}. \tag{19}$$

Let A be the nonlinear operator from $W_0^{1,p}(\Omega, w)$ into its dual $W^{-1,p'}(\Omega, w^*)$ defined by,

$$Au = -\text{div}(a(x, u, \nabla u)),$$

where $a : \Omega \times I\!R \times I\!R^N \to I\!R^N$ is a Carathéodory function satisfying the following assumptions:
(H_2) For $i = 1, \ldots, N$

$$|a_i(x, s, \xi)| \leq w_i^{\frac{1}{p}}(x)[k(x) + \sigma^{\frac{1}{p'}}|s|^{\frac{q}{p'}} + \sum_{j=1}^{N} w_j^{\frac{1}{p'}}(x)|\xi_j|^{p-1}], \tag{20}$$

$$[a(x, s, \xi) - a(x, s, \eta)](\xi - \eta) > 0 \ \text{ for all } \ \xi \neq \eta \in I\!R^N, \tag{21}$$

for a.e. x in $\Omega, s \in I\!R$, where $k(x)$ is a positive function in $L^{p'}(\Omega)$.
(H_3) $g(x, s, \xi)$ is a Carathéodory function which satisfies the following classical conditions,

$$g(x, s, \xi).s \geq 0 \tag{22}$$

and

$$|g(x, s, \xi)| \leq b(|s|)(\sum_{i=1}^{N} w_i(x)|\xi_i|^p + c(x)), \tag{23}$$

where $b : I\!R^+ \to I\!R^+$ is a positive increasing function and $c(x)$ is a positive function in $L^1(\Omega)$.

Our main result in this note is the following

Theorem 3.1. *Assume that* $(H_1) - (H_3)$ *and (7) hold and* $f \in W^{-1,p'}(\Omega)$. *Then there exists at least one solution of the following unilateral problem,*

$$\begin{cases} u \in K_\psi \ g(x, u, \nabla u) \in L^1(\Omega), \ g(x, u, \nabla u)u \in L^1(\Omega) \\ \displaystyle\int_\Omega a(x, u, \nabla u)\nabla(u - v)\, dx + \int_\Omega g(x, u, \nabla u)(u - v)\, dx \\ \qquad\qquad\qquad\qquad \leq \langle f, u - v\rangle, \\ \forall \ \ v \in K_\psi \cap L^\infty(\Omega). \end{cases} \tag{24}$$

Remark 3.1. Note that if we take $w \equiv 1$ and $\sigma \equiv 1$ in the statement of the previous theorem, then we get a new result in the nondegenerated case.

4. Proof of Theorem 3.1

For the reason of simplicity we take $f = -\text{div}F$ where $F \in \prod_{i=1}^{N} L^{p'}(\Omega, w_i^{1-p'})$.
The following lemma play an important rôle in the proof of our main result,

Lemma 4.1. [4,15] *Assume that* $(H_1), (H_2)$ *and (7) are satisfied, and let* $(u_n)_n$ *be a sequence in* $W_0^{1,p}(\Omega, w)$ *such that* $u_n \rightharpoonup u$ *weakly in* $W_0^{1,p}(\Omega, w)$ *and*

$$\int_\Omega [a(x, u_n, \nabla u_n) - a(x, u_n, \nabla u)]\nabla(u_n - u)\, dx \to 0$$

then, $u_n \to u$ *in* $W_0^{1,p}(\Omega, w)$.

We shall give the proof of Theorem in several steps.

STEP 1: Construction of the approximate unilateral problem and existence of a solution.

Let us approximate the nonlinear function g by g_n defined through (8). We consider the following approximate unilateral problem

$$\begin{cases} u_n \in K_\psi, \\ \langle Au_n, u_n - v\rangle + \int_\Omega g_n(x, u_n, \nabla u_n)(u_n - v)\, dx \leq \langle f, u_n - v\rangle \\ \forall v \in K_\psi. \end{cases} \tag{25}$$

It is easy prove that g_n satisfies the following conditions:

$$g_n(x, s, \xi)s \geq 0, \quad |g_n(x, s, \xi)| \leq |g(x, s, \xi)| \text{ and } |g_n(x, s, \xi)| \leq n$$

for a.e. x in $\Omega, s \in I\!R$ and all $\xi \in I\!R^N$.

Thus, we can define the operator $G_n : W_0^{1,p}(\Omega, w) \longrightarrow W^{-1,p'}(\Omega, w^*)$ by,

$$\langle G_n u, v\rangle = \int_\Omega g_n(x, u, \nabla u) v\, dx$$

and

$$\langle Au, v\rangle = \int_\Omega a(x, u, \nabla u)\nabla v\, dx.$$

By virtue of Hölder's inequality and due to (13) and (14), we have for all $u \in W_0^{1,p}(\Omega, w)$ and all $v \in W_0^{1,p}(\Omega, w)$,

$$\begin{aligned} \left|\int_\Omega g_n(x, u, \nabla u)v\, dx\right| &\leq \left(\int_\Omega |g_n(x, u, \nabla u)|^{q'} \sigma^{-\frac{q'}{q}}\, dx\right)^{\frac{1}{q'}} \left(\int_\Omega |v|^q \sigma\, dx\right)^{\frac{1}{q}} \\ &\leq n \left(\int_\Omega \sigma^{q'/q}\sigma^{-q'/q}\, dx\right)^{\frac{1}{q'}} \|v\|_{q,\sigma} \\ &\leq C_n\|v\|. \end{aligned} \tag{26}$$

Definition 4.1. Let Y be a reflexive Banach space. A bounded operator B from Y to its dual Y^* is called pseudo-monotone if for any sequence $u_n \in Y$ with $u_n \rightharpoonup u$ weakly in Y, and $\limsup_{n\to+\infty}\langle Bu_n, u_n - u\rangle \leq 0$, we have

$$\liminf_{n\to+\infty}\langle Bu_n, u_n - v\rangle \geq \langle Bu, u - v\rangle \quad \forall\, v \in Y.$$

Consider the operator $B_n: \ W_0^{1,p}(\Omega, w) \longrightarrow W^{-1,p'}(\Omega, w^*)$ defined as

$$\langle B_n v, w\rangle = \int_\Omega g_n(x, v, \nabla v) w \, dx + \int_\Omega a(x, v, \nabla v)\nabla w \, dx. \tag{27}$$

Lemma 4.2. *The operator B_n from K_ψ into $W^{-1,p'}(\Omega, w^*)$ is pseudomonotone. Moreover, B_n is coercive in the following sense:*

$$\frac{\langle B_n v, v - v_0\rangle}{\|v\|} \longrightarrow +\infty \quad \textit{if} \quad \|v\| \longrightarrow +\infty, v \in K_\psi,$$

where v_0 is associated to a through (7).

The proof of this Lemma will be given in Appendix 1.

Lemma 4.2 implies that the problem (25) has a solution, applying the classical result for nonlinear equations associated to pseudo-monotone operators (see [,[15] Theorem 8.2]).

STEP 2: A priori estimates on the approximate solution and convergence.

Let $k \geq \|v_0\|_\infty$ and let $\varphi_k(s) = se^{\gamma s^2}$, where $\gamma = (\frac{2b(k)}{\alpha})^2$.

An essential role will be played the following property enjoyed by $\varphi_k(s)$ (the proof is trivial):

$$\varphi_k'(s) - \frac{2b(k)}{\alpha}|\varphi_k(s)| \geq \frac{1}{2}, \quad \forall s \in I\!R. \tag{28}$$

Taking $u_n - \eta\varphi_k(T_l(u_n - v_0))$ (η small enough) as test function in (25), where $l = k + \|v_0\|_\infty$, we obtain

$$\int_\Omega a(x, u_n, \nabla u_n)\nabla T_l(u_n - v_0)\varphi_k'(T_l(u_n - v_0)) \, dx$$
$$+ \int_\Omega g_n(x, u_n, \nabla u_n)\varphi_k(T_l(u_n - v_0)) \, dx \leq \langle f, \varphi_k(T_l(u_n - v_0))\rangle.$$

Using the fact that $g_n(x, u_n, \nabla u_n)\varphi_k(T_l(u_n - v_0)) \geq 0$ on the subset $\{x \in \Omega : |u_n(x)| > k\}$, we get

$$\int_{\{|u_n - v_0| \leq l\}} a(x, u_n, \nabla u_n)\nabla(u_n - v_0)\varphi_k'(T_l(u_n - v_0)) \, dx$$
$$\leq \int_{\{|u_n| \leq k\}} |g_n(x, u_n, \nabla u_n)||\varphi_k(T_l(u_n - v_0))| \, dx$$
$$+ \int_\Omega F\nabla T_l(u_n - v_0)\varphi_k'(T_l(u_n - v_0)) \, dx.$$

Now we use (7) and (23), we can write

$$\begin{aligned}
&\alpha\int_{\{|u_n-v_0|\le l\}}\sum_{i=1}^{N}w_i(x)|\frac{\partial u_n}{\partial x_i}|^p\varphi_k'(T_l(u_n-v_0))\,dx\\
&\quad\le b(|k|)\int_\Omega\left(c(x)+\sum_{i=1}^{N}w_i(x)|\frac{\partial T_k(u_n)}{\partial x_i}|^p\right)|\varphi_k(T_l(u_n-v_0))|\,dx\\
&\quad+\int_\Omega\delta(x)\varphi_k'(T_l(u_n-v_0))\,dx+\int_{\{|u_n-v_0|\le l\}}F.\nabla u_n\varphi_k'(T_l(u_n-v_0))\,dx\\
&\quad+\int_{\{|u_n-v_0|\le l\}}F.\nabla v_0\varphi_k'(T_l(u_n-v_0))\,dx.
\end{aligned}$$

Applying the Young's inequality and the fact that $F\in\prod_{i=1}^{N}L^{p'}(\Omega,w_i^{1-p'})$, we get

$$\begin{aligned}
&\alpha\int_{\{|u_n-v_0|\le l\}}\sum_{i=1}^{N}w_i(x)|\frac{\partial u_n}{\partial x_i}|^p\varphi_k'(T_l(u_n-v_0))\,dx\\
&\quad\le b(|k|)\int_\Omega\left(c(x)+\sum_{i=1}^{N}w_i(x)|\frac{\partial T_k(u_n)}{\partial x_i}|^p\right)|\varphi_k(T_l(u_n-v_0))|\,dx\\
&\quad+\tfrac{\alpha}{2}\int_{\{|u_n-v_0|\le l\}}\sum_{i=1}^{N}w_i(x)|\frac{\partial u_n}{\partial x_i}|^p\varphi_k'(T_l(u_n-v_0))\,dx\\
&\quad+\int_\Omega\delta(x)\varphi_k'(T_l(u_n-v_0))+C_1(k)
\end{aligned}$$

where $C_1(k)$ is a positive constant depending on k.
Thanks to $\{x\in\Omega,|u_n(x)|\le k\}\subseteq\{x\in\Omega:|u_n-v_0|\le l\}$ and the fact that $c,\delta\in L^1(\Omega)$, we have

$$\int_\Omega\sum_{i=1}^{N}w_i(x)|\frac{\partial T_k(u_n)}{\partial x_i}|^p\left[\varphi_k'(T_l(u_n-v_0))-\frac{2b(k)}{\alpha}|\varphi_k(T_l(u_n-v_0))|\right]dx\le C_2(k).$$

Thus by (28), we deduce

$$\int_\Omega\sum_{i=1}^{N}w_i(x)|\frac{\partial T_k(u_n)}{\partial x_i}|^p\,dx\le 2C_2(k).\tag{29}$$

Now, we claim that,

$$\int_\Omega\sum_{i=1}^{N}w_i(x)|\frac{\partial u_n}{\partial x_i}|^p\,dx\le C.$$

Let $k \geq \|v_0\|_\infty$. Taking $v = v_0$ as a test function in (25), we get

$$\int_\Omega a(x,u_n,\nabla u_n)\nabla(u_n - v_0)\,dx + \int_\Omega g_n(x,u_n,\nabla u_n)(u_n - v_0)\,dx \leq \int_\Omega F\nabla(u_n - v_0)\,dx. \tag{30}$$

Furthermore, since $g_n(x,u_n,\nabla u_n)(u_n - v_0) \geq 0$ on the subset $\{x \in \Omega, |u_n(x)| > k\}$, the inequality (30) implies that

$$\int_\Omega a(x,u_n,\nabla u_n)\nabla(u_n - v_0)\,dx \leq 2k\int_{\{|u_n|\leq k\}} |g_n(x,u_n,\nabla u_n)|\,dx + \int_\Omega F\nabla(u_n - v_0)\,dx,$$

which gives by using (23) and Young's inequality

$$\int_\Omega a(x,u_n,\nabla u_n)\nabla(u_n - v_0)\,dx \leq 2kb(k)\left[\int_\Omega c(x)\,dx + \int_\Omega \sum_{i=1}^N w_i(x)|\frac{\partial T_k(u_n)}{\partial x_i}|^p\,dx\right] + \tfrac{\alpha}{2}\int_\Omega \sum_{i=1}^N w_i(x)|\frac{\partial u_n}{\partial x_i}|^p\,dx + C. \tag{31}$$

Using (29) and (31), we obtain

$$\int_\Omega a(x,u_n,\nabla u_n)\nabla(u_n - v_0)\,dx \leq \tfrac{\alpha}{2}\int_\Omega \sum_{i=1}^N w_i(x)|\frac{\partial u_n}{\partial x_i}|^p\,dx + C_3(k).$$

Thanks to (7), we have

$$\int_\Omega \sum_{i=1}^N w_i(x)|\frac{\partial u_n}{\partial x_i}|^p\,dx \leq C. \tag{32}$$

Then, we conclude that there exists a function $u \in W_0^{1,p}(\Omega,w)$ such that

$$\begin{array}{lll} u_n \rightharpoonup u & \text{weakly in} & W_0^{1,p}(\Omega,w), \\ u_n \to u & \text{strongly in} & L^q(\Omega,\sigma) \text{ and } a.e. \text{ in } \Omega. \end{array} \tag{33}$$

This yields, by using (20), the existence of a function $h \in \prod_{i=1}^N L^{p'}(\Omega, w_i^{1-p'})$, such that

$$a(x,u_n,\nabla u_n) \rightharpoonup h \quad \text{weakly in} \quad \prod_{i=1}^N L^{p'}(\Omega, w_i^{1-p'}). \tag{34}$$

STEP 3: Almost everywhere convergence of the gradient.
We fix $k > \|v_0\|_\infty$, and let $w_{n,k} = T_k(u_n) - T_k(u)$.

For $\eta > 0$, we consider the following function:

$$v_n = u_n - \eta\varphi_k(w_{n,k}). \tag{35}$$

We choose η such that $v_n \in K_\psi$ (see Appendix 2). As in,[7] we take v_n as a test functions in (25), we get

$$\int_\Omega a(x, u_n, \nabla u_n)\nabla w_{n,k}\varphi'_k(w_{n,k})\, dx + \int_\Omega g_n(x, u_n, \nabla u_n)\varphi_k(w_{n,k})\, dx \\ \leq \langle f, \varphi_k(w_{n,k})\rangle. \tag{36}$$

Remark that, we have

$$T_k(u_n) - T_k(u) = \begin{cases} k - T_k(u) \geq 0 & \text{if } \ u_n \geq k \\ u_n - T_k(u) \leq 0 & \text{if } \ |u_n| \leq k \\ -k - T_k(u) \leq 0 & \text{if } \ u_n \leq -k \end{cases}$$

which implies, by using (22), $g(x, u_n, \nabla u_n)\varphi_k(w_{n,k}) \geq 0$ on the set $\{x \in \Omega, |u_n(x)| > k\}$. So by (36),

$$\int_\Omega a(x, u_n, \nabla u_n)\nabla w_{n,h}\varphi'_k(w_{n,k})\, dx + \int_{\{|u_n|\leq k\}} g_n(x, u_n, \nabla u_n)\varphi_k(w_{n,k})\, dx \\ \leq \langle f, \varphi_k(w_{n,k})\rangle. \tag{37}$$

Splitting the first integral on the left hand side of (37) where $|u_n| \leq k$ and $|u_n| > k$, we can write,

$$\int_\Omega a(x, u_n, \nabla u_n)\nabla w_{n,k}\varphi'_k(w_{n,k})\, dx \\ = \int_{\{|u_n|\leq k\}} a(x, u_n, \nabla u_n)[\nabla T_k(u_n) - \nabla T_k(u)]\varphi'_k(w_{n,k})\, dx \\ + \int_{\{|u_n|>k} a(x, u_n, \nabla u_n)\nabla w_{n,k}\varphi'_k(w_{n,k})\, dx. \tag{38}$$

Now we estimate the first term in the right hand side of (38) as follows

$$\int_{\{|u_n|\leq k\}} a(x, u_n, \nabla u_n)\nabla w_{n,k}\varphi'_k(w_{n,k})\, dx \\ \geq \int_\Omega a(x, T_k(u_n), \nabla T_k(u_n))[\nabla T_k(u_n) - \nabla T_k(u)]\varphi'_k(w_{n,k})\, dx \\ - \varphi'_k(2k)\int_{\{|u_n|>k\}} \sum_{i=1}^{N} |a_i(x, T_k(u_n), 0)||\frac{\partial T_k(u)}{\partial x_i}|\, dx. \tag{39}$$

We claim that last term of the above inequality goes to zero as n tends to zero.
Indeed we have, for that $i = 1, \ldots, N \quad |a_i(x, T_k(u_n), 0)|\chi_{\{|u_n|>k\}}$ converges to $|a(x, T_k(u), 0)|\chi_{\{|u|>k\}}$ strongly in $L^{p'}(\Omega, w_i^{1-p'})$, moreover, since $|\frac{\partial T_k(u)}{\partial x_i}| \in L^p(\Omega, w_i)$, then

$$-\varphi_k'(2k)\int_{\{|u_n|>k\}} \sum_{i=1}^{N} |a_i(x, T_k(u_n), 0)||\frac{\partial T_k(u)}{\partial x_i}|\, dx = \varepsilon(n).$$

Where $\varepsilon(n)$ is a quantity (which is possible to be changed from a line to another) such that $\lim_{n\to+\infty} \varepsilon(n) = 0$ We now investigate the behaviour of the second term in the right hand side of (38) we get,

$$\int_{\{|u_n|>k\}} a(x, u_n, \nabla u_n)\nabla w_{n,k}\varphi_k'(w_{n,k})\, dx \\ \geq -\varphi_k'(2k)\int_{\{|u_n|>k\}} \sum_{i=1}^{N} |a_i(x, u_n, \nabla u_n)||\frac{\partial T_k(u)}{\partial x_i}|\, dx. \tag{40}$$

By (34) and the fact that

$$\frac{\partial T_k(u)}{\partial x_i}\chi_{\{|u_n|>k\}} \to \frac{\partial T_k(u)}{\partial x_i}\chi_{\{|u|>k\}} = 0$$

in $L^p(\Omega, w_i)$, we have by (34)

$$-\varphi_k'(2k)\int_{\{|u_n|>k\}} \sum_{i=1}^{N} |a_i(x, u_n, \nabla u_n)||\frac{\partial T_k(u)}{\partial x_i}|\, dx = \varepsilon(n) \tag{41}$$

Gathering (39), (40) and (41) we have that (38) implies:

$$\int_\Omega a(x, u_n, \nabla u_n)\nabla w_{n,k}\varphi_k'(w_{n,k})\, dx \\ \geq \int_\Omega a(x, T_k(u_n), \nabla T_k(u_n))[\nabla T_k(u_n) - \nabla T_k(u)]\varphi_k'(w_{n,k})\, dx \\ + \varepsilon(n).$$

Which implies that

$$\begin{aligned}\int_\Omega a(x,u_n,\nabla u_n)\nabla w_{n,k}\varphi_k'(w_{n,k})\,dx \\ &\geq \int_\Omega [a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u))] \\ &\times[\nabla T_k(u_n) - \nabla T_k(u)]\varphi_k'(w_{n,k})\,dx \\ &+\int_\Omega a(x,T_k(u_n),\nabla T_k(u))[\nabla T_k(u_n) - \nabla T_k(u)]\varphi_k'(w_{n,k})\,dx \\ &+\varepsilon(n).\end{aligned} \tag{42}$$

By the continuity of the Nemitskii operator (see[11]), we have for all $i = 1,\ldots,N$,

$$a_i(x,T_k(u_n),\nabla T_k(u))\varphi'(T_k(u_n) - T_k(u)) \to a_i(x,T_k(u),\nabla T_k(u))\varphi'(0)$$

strongly in $L^{p'}(\Omega,w_i^{1-p'})$, while $\frac{\partial(T_k(u_n))}{\partial x_i} \rightharpoonup \frac{\partial(T_k(u))}{\partial x_i}$ weakly in $L^p(\Omega,w_i)$. Which yields

$$\lim_{n\to\infty}\int_\Omega a(x,T_k(u_n),\nabla T_k(u))[\nabla T_k(u_n) - \nabla T_k(u))]\varphi_k'(w_{n,k})\,dx = 0. \tag{43}$$

Du to (42) and (43), we conclude that

$$\begin{aligned}\int_\Omega a(x,u_n,\nabla u_n)[\nabla T_k(u_n) - \nabla T_k(u)]\varphi'(w_{n,k})\,dx \\ &\geq \int_\Omega [a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u))] \\ &\times[\nabla T_k(u_n) - \nabla T_k(u)]\varphi'(w_{n,k})\,dx + \varepsilon(n).\end{aligned} \tag{44}$$

We now, discus the behaviour of the second integral of the left hand side of (37).

Using (23), we have

$$\begin{aligned}\left|\int_{\{|u_n|\leq k\}} g_n(x,u_n,\nabla u_n)\varphi_k(w_{n,k})\,dx\right| \\ &\leq b(k)\int_\Omega (c(x) + \sum_{i=1}^N w_i|\frac{\partial T_k(u_n)}{\partial x_i}|^p|\varphi_k(w_{n,k})|\,dx \\ &\leq b(k)\int_\Omega c(x)|\varphi_k(w_{n,k})|\,dx + \tfrac{b(k)}{\alpha}\int_\Omega \delta(x)|\varphi_k(w_{n,k})| \\ &+\tfrac{b(k)}{\alpha}\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n)|\varphi_k(w_{n,k})|\,dx \\ &-\tfrac{b(k)}{\alpha}\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\nabla v_0|\varphi_k(w_{n,k})|\,dx.\end{aligned}$$

From (20) and (29), there exists a function $h_k \in \prod_{i=1}^{N} L^{p'}(\Omega, w_i^{1-p'})$ such that

$$a(x, T_k(u_n), \nabla T_k(u_n)) \rightharpoonup h_k \quad \text{weakly in} \quad \prod_{i=1}^{N} L^{p'}(\Omega, w_i^{1-p'}) \tag{45}$$

Moreover, since $\nabla v_0 |\varphi_k(w_{n,k})| \to 0$ in $\prod_{i=1}^{N} L^p(\Omega, w_i)$ as n tend to infinity, the third term of the right hand side of the last inequality tend to zero as n tend to infinity, hence

$$\begin{aligned}
&\left| \int_{\{|u_n| \le k\}} g_n(x, u_n, \nabla u_n) \varphi_k(w_{n,k}) \, dx \right| \\
&\qquad \le \tfrac{b(k)}{\alpha} \int_\Omega a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(u_n) |\varphi_k(w_{n,k})| \, dx \\
&\qquad + b(k) \int_\Omega c(x) |\varphi_k(w_{n,k}| \, dx + \tfrac{b(k)}{\alpha} \int_\Omega \delta(x) |\varphi_k(w_{n,k})| \, dx + \varepsilon(n).
\end{aligned} \tag{46}$$

Next we estimate the first term in the right hand side of (46) as follows:

$$\begin{aligned}
&\tfrac{b(k)}{\alpha} \int_\Omega a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(u_n) |\varphi_k(w_{n,k})| \, dx \\
&\quad = \tfrac{b(k)}{\alpha} \int_\Omega [a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u)] \\
&\quad \times [\nabla T_k(u_n) - \nabla T_k(u)] |\varphi_k(w_{n,k})| \, dx \\
&\quad + \tfrac{b(k)}{\alpha} \int_\Omega a(x, T_k(u_n), \nabla T_k(u)) [\nabla T_k(u_n) - \nabla T_k(u))] |\varphi_k(w_{n,k})| \, dx \\
&\quad + \tfrac{b(k)}{\alpha} \int_\Omega a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(u) |\varphi_k(w_{n,k})| \, dx.
\end{aligned} \tag{47}$$

By Lebesgue's Theorem, we deduce that

$$\nabla T_k(u) |\varphi_k(w_{n,k})| \to \nabla T_k(u) |\varphi_k(0)| = 0 \quad \text{strongly in} \quad \prod_{i=1}^{N} L^p(\Omega, w_i).$$

Which and using (45) implies that the third term of (47) tends to 0 as $n \to \infty$.

On the other side reasoning as in (43), the second term of (47) tends to 0 as $n \to \infty$.

From (46) and (47), we obtain

$$\begin{aligned}
&\left| \int_{\{|u_n|\le k\}} g_n(x,u_n,\nabla u_n)\varphi_k(w_{n,k})\,dx \right| \\
&\quad \le \int_\Omega [a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u))] \\
&\quad \times [\nabla T_k(u_n) - \nabla T_k(u)]|\varphi_k(w_{n,k})|\,dx \\
&\quad + b(k)\int_\Omega c(x)|\varphi_k(w_{n,k})|\,dx + \tfrac{b(k)}{\alpha}\int_\Omega \delta(x)|\varphi_k(w_{n,k})|\,dx + \varepsilon(n).
\end{aligned} \tag{48}$$

Combining (37), (44) and (48), we obtain

$$\begin{aligned}
&\int_\Omega [a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u))] \\
&\qquad \times [\nabla T_k(u_n) - \nabla T_k(u)](\varphi_k'(w_{n,k}) - \tfrac{b(k)}{\alpha}|\varphi_k(w_{n,k})|)\,dx \\
&\qquad \le b(k)\int_\Omega c(x)|\varphi_k(w_{n,k})|\,dx + \tfrac{b(k)}{\alpha}\int_\Omega \delta(x)|\varphi_k(w_{n,k})|\,dx \\
&\qquad + \langle f, \varphi_k(w_{n,k})\rangle + \varepsilon(n).
\end{aligned} \tag{49}$$

Therefore (28) implies

$$\begin{aligned}
&\int_\Omega [a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u))] \\
&\qquad \times [\nabla T_k(u_n) - \nabla T_k(u)]\,dx \\
&\qquad \le 2b(k)\int_\Omega c(x)|\varphi_k(w_{n,k})|\,dx + 2\tfrac{b(k)}{\alpha}\int_\Omega \delta(x)|\varphi_k(w_{n,k})|\,dx \\
&\qquad + \langle f, \varphi_k(w_{n,k})\rangle + \varepsilon(n).
\end{aligned} \tag{50}$$

Now, since $c, \delta \in L^1(\Omega)), \varphi_k(w_{n,k}) \rightharpoonup 0$ weakly in $W_0^{1,p}(\Omega, w)$ as $n \to \infty$, all the terms of the right hand side of the last inequality tends to 0 as $n \to +\infty$.
This implies that

$$\lim_{n\to\infty} \int_\Omega [a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u))][\nabla T_k(u_n) - \nabla T_k(u)]\,dx = 0.$$

Finally, Lemma 4.1 implies that

$$T_k(u_n) \to T_k(u) \;\text{ strongly in }\; W_0^{1,p}(\Omega, w) \;\; \forall k > 0. \tag{51}$$

Since k arbitrary, we have for a subsequence

$$\nabla u_n \to \nabla u \;\text{ a.e. in }\; \Omega. \tag{52}$$

Indeed, we show that (as in[8]) ∇u_n converge to ∇u in measure (and therefore, we can always assume that the convergence is a.e. after passing to a

suitable subsequence).
Let $k>0$ large enough, we have

$$\begin{aligned} kmeas(\{|u_n|>k\}\cap B_R) &= \int_{\{|u_n|>k\}\cap B_R} |T_k(u_n)|\,dx \le \int_{B_R} |T_k(u_n)|\,dx \\ &\le \left(\int_\Omega |u_n|^p w_0\,dx\right)^{\frac{1}{p}} \left(\int_{B_R} w_0^{1-p'}\,dx\right)^{\frac{1}{q'}} \\ &\le c_0 \left(\int_\Omega \sum_{i=1}^N |\frac{\partial u_n}{\partial x_i}|^p w_i(x)\,dx\right)^{\frac{1}{p}} \\ &\le c_1 \end{aligned}$$

where $B_R=\{x\in\Omega; |x|\le R\}$. Which implies that

$$meas(\{|u_n|>k\}\cap B_R) \le \frac{c_1}{k} \quad \forall k>1. \tag{53}$$

same, since $u\in W_0^{1,p}(\Omega,w)$, we get

$$meas(\{|u|>k\}\cap B_R) \le \frac{c_2}{k} \quad \forall k>1. \tag{54}$$

We have, for every $\delta>0$,

$$\begin{aligned} meas(\{|\nabla u_n - \nabla u|>\delta\}) &\le meas(\{|u_n|>k\}) + meas(\{|u|>k\}) \\ &+meas(\{|\nabla T_k(u_n)-\nabla T_k(u)|>\delta\}) \\ &\le meas(\{|u_n|>k\}\cap B_R) + meas(\{|u|>k\}\cap B_R) \\ &+2meas(\{|x|>R\}) + meas(\{|\nabla T_k(u_n)-\nabla T_k(u)|>\delta\}). \end{aligned} \tag{55}$$

Since $T_k(u_n)$ converge strongly in $W_0^{1,p}(\Omega,w)$, we can assume that $\nabla T_k(u_n)$ converge to $\nabla T_k(u)$ in measure in Ω.

Let $\varepsilon>0$, for R large enough, by (53), (54) and (55), there exists some $n_0(k,R,\delta,\varepsilon)>0$ such that $meas(\{|\nabla u_n-\nabla u|>\delta\})<\varepsilon$ for all $n,m\ge n_0(k,R,\delta,\varepsilon)$. This concludes the proof of (52).
Which yields

$$\begin{cases} a(x,u_n,\nabla u_n)\to a(x,u,\nabla u) \text{ a.e. in } \Omega \\ g_n(x,u_n,\nabla u_n)\to g(x,u,\nabla u) \text{ a.e. in } \Omega. \end{cases} \tag{56}$$

Step 4: Equi-integrability of the nonlinearities.
We need to prove that

$$g_n(x,u_n,\nabla u_n)\to g(x,u,\nabla u) \text{ strongly in } L^1(\Omega), \tag{57}$$

in particular it is enough to prove the equi-integrable of $g_n(x,u_n,\nabla u_n)$. To this purpose. We consider the function $T_1(u_n-v_0-T_h(u_n-v_0))$ (with

$h \geq \|v_0\|_\infty$). This function can be write as follows

$$T_1(u_n - v_0 - T_h(u_n - v_0)) = \begin{cases} 0 & \text{if } |u_n - v_0| \leq h \\ u_n - v_0 - h & \text{if } h \leq u_n - v_0 \leq h+1 \\ u_n - v_0 + h & \text{if } -h-1 \leq u_n - v_0 \leq -h \\ 1 & \text{if } u_n - v_0 \geq h+1 \\ -1 & \text{if } u_n - v_0 \leq -h-1 \end{cases}$$

which implies that

$$g_n(x, u_n, \nabla u_n) T_1(u_n - v_0 - T_h(u_n - v_0)) \geq 0 \tag{58}$$

and

$$g_n(x, u_n, \nabla u_n) T_1(u_n - v_0 - T_h(u_n - v_0)) = |g_n(x, u_n, \nabla u_n)| \tag{59}$$

on the set $\{x \in \Omega; |u_n - v_0| \geq h+1\}$.

Now, we take $u_n - T_1(u_n - v_0 - T_h(u_n - v_0))$ as a test function in (25), we obtain by using (58) and (59)

$$\begin{aligned} \int_{\{h \leq |u_n - v_0| \leq h+1\}} & a(x, u_n, \nabla u_n) \nabla(u_n - v_0)\, dx \\ & + \int_{\{|u_n - v_0| > h+1\}} |g_n(x, u_n, \nabla u_n)|\, dx \\ & \leq \int_{\{h \leq |u_n - v_0| \leq h+1\}} F \nabla(u_n - v_0)\, dx \end{aligned}$$

using the Young's inequality and (7), we deduce

$$\begin{aligned} \int_{\{|u_n - v_0| > h+1\}} |g_n(x, u_n, \nabla u_n)|\, dx \leq c \sum_{i=1}^N \int_{\{|u_n - v_0| > h\}} |F_i|^{p'} w_i^{1-p'}\, dx \\ + \int_{\{|u_n - v_0| > h\}} (|F \nabla v_0| + \delta(x))\, dx \end{aligned} \tag{60}$$

Let $\varepsilon > 0$, since the functions $|F_i|^{p'} w_i^{1-p'}, |F \nabla v_0|$ and δ belongs to $L^1(\Omega)$ then there exists $h(\varepsilon) \geq 1$ such that

$$\int_{\{|u_n - v_0| > h(\varepsilon)\}} |g(x, u_n, \nabla u_n)|\, dx < \varepsilon/2. \tag{61}$$

For any measurable subset $E \subset \Omega$, we have

$$\begin{aligned} & \int_E |g_n(x, u_n, \nabla u_n)|\, dx \leq \int_E b(h(\varepsilon) + \|v_0\|_\infty)(c(x) \\ & + \sum_{i=1}^N w_i |\tfrac{\partial T_{h(\varepsilon) + \|v_0\|_\infty}(u_n)}{\partial x_i}|^p)\, dx + \int_{\{|u_n - v_0| > h(\varepsilon)\}} |g(x, u_n, \nabla u_n)|\, dx. \end{aligned} \tag{62}$$

In view of (51) there exists $\eta(\varepsilon) > 0$ such that

$$\int_E b(h(\varepsilon) + \|v_0\|_\infty)(c(x) + \sum_{i=1}^{N} w_i |\frac{\partial T_{h(\varepsilon)+\|v_0\|_\infty}(u_n)}{\partial x_i}|^p)\, dx < \varepsilon/2 \tag{63}$$

for all E such that meas $E < \eta(\varepsilon)$.
Finally, combining (61), (62) and (63), one easily has

$$\int_E |g_n(x, u_n, \nabla u_n)|\, dx < \varepsilon \quad \text{for all } E \text{ such that meas } E < \eta(\varepsilon),$$

which implies (57)

Step 5: Passing to the limit.

We take $v \in K_\psi \cap L^\infty(\Omega)$ as test function in (25), we can write

$$\begin{aligned}\int_\Omega a(x, u_n, \nabla u_n)\nabla(u_n - v)\, dx + \int_\Omega g_n(x, u_n, \nabla u_n)(u_n - v)\, dx \\ \le \langle f, u_n - v\rangle.\end{aligned} \tag{64}$$

This implies

$$\begin{aligned}\int_\Omega a(x, u_n, \nabla u_n)\nabla(u_n - v_0)\, dx + \int_\Omega a(x, u_n, \nabla u_n)\nabla(v_0 - v)\, dx \\ + \int_\Omega g_n(x, u_n, \nabla u_n)(u_n - v)\, dx \\ \le \langle f, u_n - v\rangle.\end{aligned} \tag{65}$$

By Fatou's Lemma and the fact that

$$a(x, u_n, \nabla u_n) \rightharpoonup a(x, u, \nabla u)$$

weakly in $\prod_{i=1}^{N} L^{p'}(\Omega, w_i^{1-p'})$ one easily sees that

$$\begin{aligned}\int_\Omega a(x, u, \nabla u)\nabla(u - v_0)\, dx + \int_\Omega a(x, u, \nabla u)\nabla(v_0 - v)\, dx \\ + \int_\Omega g(x, u, \nabla u)(u - v)\, dx \\ \le \langle f, u - v\rangle.\end{aligned} \tag{66}$$

Hence

$$\begin{aligned}\int_\Omega a(x, u, \nabla u)\nabla(u - v)\, dx + \int_\Omega g(x, u, \nabla u)(u - v)\, dx \\ \le \langle f, u - v\rangle.\end{aligned} \tag{67}$$

On the other hand, for h large enough we have

$$\int_\Omega |g_n(x,u_n,\nabla u_n)|\,dx \le \int_\Omega b(h+\|v_0\|_\infty)(c(x)+\sum_{i=1}^N w_i|\frac{\partial T_{h+\|v_0\|_\infty}(u_n)}{\partial x_i}|^p)\,dx + \int_{\{|u_n-v_0|>h\}} |g(x,u_n,\nabla u_n)|\,dx. \tag{68}$$

combining (61), (68) and the fact that u_n bounded in $W_0^{1,p}(\Omega,w)$, we get

$$\int_\Omega |g_n(x,u_n,\nabla u_n)|\,dx \le c_1 \tag{69}$$

taking also v_0 as a test function in (25), we have

$$\int_\Omega g_n(x,u_n,\nabla u_n)u_n\,dx \le c_2 \tag{70}$$

by (69), (70) and Fatou's Lemma we deduce

$$g(x,u,\nabla u)\in L^1(\Omega), g(x,u,\nabla u)u \in L^1(\Omega).$$

This proves Theorem 3.1.

5. Appendix

Appendix 1: Proof of Lemma 4.2
From Hölder's inequality, the growth condition (20) we can show that A is bounded, and by using (26), we have B_n bounded. The coercivity follows from (7), (22) and (26). it remain to show that B_n is pseudo-monotone. Let a sequence $(u_k)\in W_0^{1,p}(\Omega,w)$ such that

$$u_k \rightharpoonup u \text{ weakly in } W_0^{1,p}(\Omega,w),$$

$$\limsup_{k\to+\infty}\langle B_n u_k, u_k-u\rangle \le 0. \tag{71}$$

Let $v\in W_0^{1,p}(\Omega,w)$, we will prove that

$$\liminf_{k\to+\infty}\langle B_n u_k, u_k-v\rangle \ge \langle B_n u, u-v\rangle.$$

Since $(u_k)_k$ is bounded in $W_0^{1,p}(\Omega,w)$, by (20) we deduce that $(a(x,u_k,\nabla u_k))_k$ is bounded in $\prod_{i=1}^N L^{p'}(\Omega,w_i^{1-p'})$, then there exists a function $h\in\prod_{i=1}^N L^{p'}(\Omega,w_i^{1-p'})$ such that

$$a(x,u_k,\nabla u_k)\rightharpoonup h \text{ weakly in } \prod_{i=1}^N L^{p'}(\Omega,w_i^{1-p'}),$$

similarly, it is easy to see that $(g_n(x,u_k,\nabla u_k))_k$ is bounded in $L^{q'}(\Omega,\sigma^{1-q'})$, then there exists a function $\rho_n \in L^{q'}(\Omega,\sigma^{1-q'})$ such that

$$g_n(x,u_k,\nabla u_k) \rightharpoonup \rho_n \quad \text{weakly in} \quad Ł^{q'}(\Omega,\sigma^{1-q'}).$$

It is clear that, (15)

$$\liminf_{k\to+\infty}\langle B_n u_k, u_k - v\rangle = \liminf_{k\to+\infty}\int_\Omega a(x,u_k,\nabla u_k)\nabla u_k\,dx - \int_\Omega h\nabla v\,dx + \int_\Omega \rho_n(u-v)\,dx. \tag{72}$$

On the other hand, by condition (21), we have

$$\int_\Omega (a(x,u_k,\nabla u_k) - a(x,u_k,\nabla u))(\nabla u_k - \nabla u)\,dx \geq 0$$

which implies that

$$\int_\Omega a(x,u_k,\nabla u_k)\nabla u_k\,dx \geq -\int_\Omega a(x,u_k,\nabla u)\nabla u\,dx + \int_\Omega a(x,u_k,\nabla u_k)\nabla u\,dx + \int_\Omega a(x,u_k,\nabla u)\nabla u_k\,dx,$$

hence

$$\liminf_{k\to\infty}\int_\Omega a(x,u_k,\nabla u_k)\nabla u_k\,dx \geq \int_\Omega h\nabla u\,dx. \tag{73}$$

Combining (72) and (73), we get

$$\liminf_{k\to+\infty}\langle B_n u_k, u_k - v\rangle \geq \int_\Omega h\nabla(u-v)\,dx + \int_\Omega \rho_n(u-v)\,dx. \tag{74}$$

Now, since v is arbitrary and $\lim\limits_{k\to+\infty}\int_\Omega g_n(x,u_k,\nabla u_k)(u_k-u)\,dx = 0$, we have by using (71) and (74)

$$\lim_{k\to+\infty}\int_\Omega a(x,u_k,\nabla u_k)\nabla(u_k-u)\,dx = 0$$

we deduce that

$$\lim_{k\to+\infty}\int_\Omega (a(x,u_k,\nabla u_k) - a(x,u_k,\nabla u))\nabla(u_k-u)\,dx = 0.$$

In view of Lemma 4.1, we have $\nabla u_k \to \nabla u$ a.e. in Ω, which with (74) yields

$$\liminf_{k\to+\infty}\langle B_n u_k, u_k - v\rangle \geq \langle B_n u, u - v\rangle.$$

Thus complete the proof of Lemma 4.2.

Appendix 2: We claim that $v_n \in K_\psi$ (v_n defined in (35))

We have $e^{\gamma w_{n,k}^2} \leq c_k$, let $\eta = \frac{1}{c_k}$ Since $v_n = u_n - \eta\varphi_k(w_{n,k})$, we remark that

$$v_n \geq \begin{cases} u_n & \text{if } \ w_{n,k} \leq 0 \\ u_n - w_{n,k} & \text{if } \ w_{n,k} \geq 0 \end{cases}$$

then it suffices to prove that $u_n - w_{n,k} \geq \psi$ we have

$$u_n - w_{n,k} = \begin{cases} T_k(u) & \text{if } \ |u_n| \leq k \\ u_n - k + T_k(u) & \text{if } \ u_n \geq k \\ u_n + k + T_k(u) & \text{if } \ u_n \leq -k \end{cases}$$

which implies that

$$u_n - w_{n,k} \geq \begin{cases} T_k(u) & \text{if } \ |u_n| \leq k \\ T_k(u) & \text{if } \ u_n \geq k \\ u_n & \text{if } \ u_n \leq -k \end{cases}$$

since $u \in K_\psi,\ k \geq \|v_0\|_\infty$ then $T_k(u) \geq \psi$, which implies that $u_n - w_{n,k} \geq \psi$. Finally since $v_n \in W_0^{1,p}(\Omega, w)$ then $v_n \in K_\psi$.

References

1. L. Aharouch and Y. Akdim, *Existence of solution of degenerated unilateral Problems with L^1 data,* Annale Mathématique Blaise Pascal Vol. **11** (2004) pp 47-66.
2. L. Aharouch, Y. Akdim and E. Azroul, *Quasilinear degenerate elliptic unilateral problems,* AAA 2005:1 (2005) 11-31. DOI: 10.1155/AAA.2005.11
3. Y. Akdim, E. Azroul and A. Benkirane, *Existence Results for Quasilinear Degenerated Equations via Strong Convergence of Truncations,* Revista Matemática Complutence, Madrid, (2004), **17**; N 2, 359-379.
4. Y. Akdim, E. Azroul and A. Benkirane, *Existence of solution for Quasilinear Degenerated Unilateral Problems,* Annale Mathématique Blaise Pascal Vol. **10** (2003) pp 1-20.
5. P. Bénilan, L. Boccardo, T. Gallouet, R. Gariepy, M. Pierre and J. L. Vázquez, *An L^1-theory of existence and uniqueness of nonlinear elliptic equations,* Ann. Scuola Norm. Sup. Pisa **22** (1995), 240-273.
6. L. Boccardo, F. Murat, and J. P. Puel , *Existence of bounded solutions for nonlinear elliptic unilateral problems,* Ann. Mat. Pura e Appl., **152** (1988) 183-196.
7. L. Boccardo, T. Gallouët and F. Murat, *A unified presentation of two existence results for problems with natural growth,* in Progress in PDE, the Metz surveys 2, M. Chipot editor, Research in Mathematics, Longman, **296** (1993), 127-137.

8. G. Dalmaso, F. Murat, L. Orsina and A. Prignet, *Renormalized solutions of elliptic equations with general measure data,* Ann. Scuola Norm. Sup Pisa Cl. Sci 12 4 (1999) 741-808.
9. P. Drabek, F. Nicolosi, *Existence of bounded solutions for some degenerated quasilinear elliptic equations,* Ann. Math. Pura Appl. (4) **165** (1993), 217-238.
10. P. Drabek, A. Kufner and V. Mustonen, *Pseudo-monotonicity and degenerated or singular elliptic operators,* Bull. Austral. Math. Soc. **58** (1998), 213-221.
11. P. Drabek, A. Kufner and F. Nicolosi, *Non linear elliptic equations, singular and degenerate cases,* University of West Bohemia, (1996).
12. D. Gilbarg and N.S. Trudinger, *Elliptic partial differential equations of second order,* Springer-Verlag, Berlin, (1977).
13. A. Porretta, *Existence for elliptic equations in L^1 having lower order terms with natural growth,*Portugal. Math. **57** (2000), 179-190.
14. A. Elmahi and D. Meskine , *Unilateral elleptic problems in L^1 with natural growth terms,* Journal of Nonlinear and Convex Analysis, **5** N. 1, (2004), 97-112
15. J. Lions, *Quelques méthodes de résolution des problèmes aux limites non linéaires,*Dunod, Paris (1969).
16. B. Opic and A. Kufner, *Hardy-type inequalities,* Pitman Research Notes in Mathematics Series vol. 219, Longman Scientific & Technical, Harlow, 1990, ISBN 0-582-05198-3.

Existence and multiplicity results for some $p(x)$-Laplacian Neumann problems

M. Bendahmane

Al-Imam University, Faculty of Sciences, Department of Mathematics
P. O. Box 90950, Riyadh 11623, Saudi Arabia
E-mail: mostafab@ing-mat.udec.cl

M. Chrif

Department of Mathematics, Faculty of Sciences, Atlas-Fes
Fes, 3000, Morocco
E-mail: moussachrif@yahoo.fr

S. El Manouni

Department of Mathematics, Faculty of Sciences, Al-Imam University
P. O. Box 90950, Riyadh, 11623, Saudi Arabia
E-mail: samanouni@imamu.edu.sa

In this paper, using an equivalent variational approach to a recent Ricceri's three critical points theorem,[13] we obtain the existence of at least three non-trivial solutions of a Neumann problem for elliptic equations with variable exponents.

Keywords: $p(x)$-Laplacian; Elliptic equations; Neumann conditions; Three critical points theorem.

1. Introduction

Let Ω be a bounded domain of $\mathbb{R}^N, N \geq 2$ with a smooth boundary $\partial\Omega$. We consider the following Neumann problem for the corresponding elliptic problem

$$(P_1)\begin{cases} -div(|\nabla u|^{p(x)-2}\nabla u) + |u|^{p(x)-2}u = \lambda f(x,u) + \mu g(x,u) & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega \end{cases}$$

where $p \in C(\overline{\Omega}), p(x) > 1$ for every $x \in \Omega$. f is a C^1-function on $\Omega \times [0,\infty)$ satisfying

$$|f(x,t)| \leq c_1\, t^{\alpha(x)-1} + c_2 \quad \forall (x,t) \tag{1.1}$$

for some $\alpha \in C_+(\overline{\Omega})$ with

$$\alpha^+ < p^-, \tag{1.2}$$

where

$$h^+ = \sup_{x\in\overline{\Omega}} h(x) \text{ and } h^- = \inf_{x\in\overline{\Omega}} h(x) \ \forall h \in C_+(\overline{\Omega}).$$

$$f(x,t) < 0, \text{ for all } t \in]0,1[, \tag{1.3}$$

$$f(x,t) > M, \text{ for all } t > t_0 \tag{1.4}$$

where M is a positive constant and $t_0 > 1$. The function g is assumed to be a measurable function with respect to x in Ω for every t in $\mathbb{R}$, and is a C^1-function with respect to t in $\mathbb{R}$ for almost every x in Ω and satisfies

$$\sup_{|t|\leq s} |g(x,t)| \in L^1(\Omega) \tag{1.5}$$

for all $s > 0$. Here Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $div(|\nabla u|^{p(x)-2}\nabla u)$ is the $p(x)-$Laplacian, the generalization of the classical $p-$Laplacian operator, and ν is the outward unit normal to $\partial\Omega$.

Recently, elliptic equations with variable exponents have been extensively investigated and have received much attention. They have been the subject of recent developments in nonlinear elasticity theory and electrorheological fluids dynamics Ružicka.[15] In that context, let us mention that there appeared a series of papers on problems which lead to spaces with variable exponent. We refer the reader to Fan et al.,[8,9] Ružicka[16] and the references therein.

Let us point out that when $p(x) = p =$ constant, there is a large literature which deal with problems involving the p-Laplacian with Dirichlet boundary conditions for elliptic equations in bounded or unbounded domains and we do not need here to cite them since the reader may easily reach such papers.

Note also that many papers deal with problems related to the p-Laplacian with Neumann conditions. We can cite, among others, the articles Anello et al.[1] and Bonanno et al.[4] and the references therein for details. The case of $p(x)-$Laplacian with Neumann conditions has been studied by Dai,[5] Mihăilescu[10] and Shi and Ding,[17] all the authors used a past result of Ricceri.[12]

Finally, it would be interesting at this stage to refer the reader to the recent work Manouni et al.[6] where the authors established the existence of at least three nontrivial solutions for elliptic systems involving the p-Laplacian with Neumann boundary conditions.

Our objective is to study the Neumann problem for such an equation of the type (P_1). Precisely, based on a recent result due to Ricceri,[13] we are interested in the existence and multiplicity of weak nontrivial solutions for the problem (P_1) in the Sobolev space $W^{1,p(x)}(\Omega)$.

Along this paper we fix $p^- > N$. Recall that a weak solution of system (P_1) is any $u \in W^{1,p(x)}(\Omega)$ such that

$$\int_\Omega (|\nabla u|^{p(x)-2}\nabla u \nabla\varphi + |u|^{p(x)-2}u\varphi)\,dx = \int_\Omega (\lambda f(x,u) + \mu g(x,u))\varphi\,dx$$

for all $\varphi \in W^{1,p(x)}(\Omega)$.

Remark 1.1. Let us remark that (1.2) and (1.5) guarantees that integrals given in the right side are well defined.

In particular, we will consider the following Neumann problem

$$(P_1)' \begin{cases} -\Delta_{p(x)}u + |u|^{p(x)-2}u = \lambda(|u|^{\alpha(x)-2}u - 1) + \mu|u|^{\gamma(x)-2}u & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega, \end{cases}$$

where

$$\gamma \in C_+(\overline{\Omega}),\ \gamma > 1 \tag{1.6}$$

and α satisfies

$$\alpha^+ < p^-. \tag{1.7}$$

To prove the existence of at least three weak solutions for each of the given problems $(P_1)'$ and (P_1), we will use the following result proved in Ricceri[13] that, on the basis of Bonanno,[2] can be equivalently stated as follows

Theorem 1.1. *Let X be a reflexive real Banach space; $\Phi : X \longrightarrow I\!R$ is bounded on each bounded subset of X, continuously Gâteaux differentiable and sequentially weakly lower semi-continuous functional whose Gâteaux derivative admits a continuous inverse on X^*; $J : X \longrightarrow I\!R$ a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact. Moreover, assume that*

(i) $$\lim_{\|u\|\longrightarrow\infty} (\Phi(u) + \lambda J(u)) = +\infty \text{ for all } \lambda \in\,]0, +\infty[$$

and that there are $r \in I\!R, u_0, u_1 \in X$ such that

(ii) $\Phi(u_0) < r < \Phi(u_1)$

$$(iii) \quad \inf_{u\in\Phi^{-1}(]-\infty,r])} J(u) > \frac{(\Phi(u_1)-r)J(u_0)+(r-\Phi(u_0))J(u_1)}{\Phi(u_1)-\Phi(u_0)}.$$

Then, there exist an open interval A of $]0,+\infty[$ and a positive real number t such that for every $\lambda \in A$, and every continuously Gâteaux differentiable functional $\Psi : X \longrightarrow I\!R$ with compact derivative, there exists $\delta > 0$ such that for each $\mu \in [0,\delta]$, the equation

$$\Phi^{'}(u) + \lambda J^{'}(u) + \mu\Psi^{'}(u) = 0$$

has at least three solutions in X whose norms are less than t.

2. Preliminaries

We list some well known definitions and basic properties and recall some background facts concerning generalized Lebesgue-Sobolev spaces $L^{p(x)}(\Omega)$, $W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$, where Ω is an open subset of $I\!R^N$ and introduce some notations used below. For more details about these spaces, we refer the reader to the book of Musielak[11] and the papers of Kováčik et al.[7] and Fan et al.[8,9]

Set

$$L^\infty_+(\Omega) = \{h;\ h \in L^\infty(\Omega), \operatorname*{ess\,inf}_{x\in\Omega} h(x) > 1 \text{ for all } x \in \overline{\Omega}\}.$$

For any $h \in L^\infty_+(\Omega)$ we define

$$h^+ = \operatorname*{ess\,sup}_{x\in\Omega} h(x) \quad \text{and} \quad h^- = \operatorname*{ess\,inf}_{x\in\Omega} h(x).$$

For any $p(x) \in L^\infty_+(\Omega)$, we define the variable exponent Lebesgue space

$$L^{p(x)}(\Omega) = \{u : u \text{is a measurable real-valued function such that } \int_\Omega |u(x)|^{p(x)}\,dx < \infty\}.$$

We define a norm, the so-called *Luxemburg norm*, on this space by the formula

$$|u|_{p(x)} = \inf\Big\{\mu > 0;\ \int_\Omega \Big|\frac{u(x)}{\mu}\Big|^{p(x)}\,dx \le 1\Big\}.$$

Variable exponent Lebesgue spaces resemble classical Lebesgue spaces in many respects: they are Banach spaces (Theorem 2.5, Kováčik[7]), the Hölder inequality holds (Theorem 2.1, Kováčik[7]), they are reflexive if and only if $1 < p^- \le p^+ < \infty$ (Corollary 2.7, Kováčik[7]) and continuous functions are dense if $p^+ < \infty$ (Theorem 2.11, Kováčik[7]). The inclusion between

Lebesgue spaces also generalizes naturally (Theorem 2.8, Kováčik[7]): if $0 < |\Omega| < \infty$ and r_1, r_2 are variable exponents so that $r_1(x) \leq r_2(x)$ almost everywhere in Ω then there exists the continuous embedding $L^{r_2(x)}(\Omega) \hookrightarrow L^{r_1(x)}(\Omega)$, whose norm does not exceed $|\Omega| + 1$. We denote by $L^{p'(x)}(\Omega)$ the conjugate space of $L^{p(x)}(\Omega)$, where $1/p(x) + 1/p'(x) = 1$. For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$ the Hölder type inequality

$$\Big| \int_\Omega uv\, dx \Big| \leq \Big(\frac{1}{p^-} + \frac{1}{p'^-}\Big) |u|_{p(x)} |v|_{p'(x)}$$

holds true. An important role in manipulating the generalized Lebesgue-Sobolev spaces is played by the *modular* of the $L^{p(x)}(\Omega)$ space, which is the mapping

$$\rho_{p(x)} : L^{p(x)}(\Omega) \to I\!R$$

defined by

$$\rho_{p(x)}(u) = \int_\Omega |u|^{p(x)}\, dx.$$

If $u \in L^{p(x)}(\Omega)$ and $p^+ < \infty$ then the following relations hold

$$|u|_{p(x)} > 1 \ \Rightarrow \ |u|_{p(x)}^{p^-} \leq \rho_{p(x)}(u) \leq |u|_{p(x)}^{p^+};$$

$$|u|_{p(x)} < 1 \ \Rightarrow \ |u|_{p(x)}^{p^+} \leq \rho_{p(x)}(u) \leq |u|_{p(x)}^{p^-};$$

$$|u_n - u|_{p(x)} \to 0 \ \Leftrightarrow \ \rho_{p(x)}(u_n - u) \to 0.$$

We also consider the weighted variable exponent Lebesgue spaces. We define also the variable Sobolev space

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) : \ |\nabla u| \in L^{p(x)}(\Omega)\}.$$

On $W^{1,p(x)}(\Omega)$ we may consider one of the following equivalent norms

$$\|u\|_{p(x)} = |u|_{p(x)} + |\nabla u|_{p(x)}$$

or

$$\|u\| = \inf \Big\{\mu > 0;\ \int_\Omega \Big(\Big|\frac{\nabla u(x)}{\mu}\Big|^{p(x)} + \Big|\frac{u(x)}{\mu}\Big|^{p(x)}\Big)\, dx \leq 1\Big\}.$$

Set

$$I_{p(x)}(u) = \int_\Omega \big(|\nabla u|^{p(x)} + |u|^{p(x)}\big)\, dx.$$

For all $u \in W^{1,p(x)}(\Omega)$ the following relations hold

$$\|u\| > 1 \Rightarrow \|u\|^{p^-} \leq I_{p(x)}(u) \leq \|u\|^{p^+}; \tag{2.1}$$

$$\|u\| < 1 \Rightarrow \|u\|^{p^+} \leq I_{p(x)}(u) \leq \|u\|^{p^-}. \tag{2.2}$$

Finally, we remember some embedding results regarding variable exponent Lebesgue-Sobolev spaces. For the continuous embedding between variable exponent Lebesgue-Sobolev spaces we refer to (Theorem 1.1, Fan et al.[8]): if $p : \Omega \to I\!R$ is Lipschitz continuous and $p^+ < N$, then for any $q \in L^\infty_+(\Omega)$ with $p(x) \leq q(x) \leq \frac{Np(x)}{N-p(x)}$, there is a continuous embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$. In what concerns if $N < p^- = \operatorname{ess\,inf}_{x\in\overline{\Omega}} p(x) \leq p(x)$ for any $x \in \overline{\Omega}$, by Theorem 2.2 in Fan et al.[9] we deduce that $W^{1,p(x)}(\Omega)$ is continuously embedded in $W^{1,p^-}(\Omega)$. Since $N < p^-$ it follows that $W^{1,p^-}(\Omega)$ is compactly embedded in $C(\overline{\Omega})$. Thus, we obtain that $W^{1,p(x)}(\Omega)$ is continuously embedded in $C(\overline{\Omega})$.

From now on, E is the space $W^{1,p(x)}(\Omega)$ with the norm

$$\|u\| = \left(\int_\Omega |\nabla u|^{p(x)} + |u|^{p(x)}\, dx\right)^{1/p(x)},$$

which is clearly equivalent to the usual one, on the space $C(\overline{\Omega})$ we consider the norm $\|u\|_\infty = \sup_{x\in\bar{\Omega}} |u(x)|$. When $p^- > N$, Sobolev's Theorem implies that E is compactly embedded in $C(\overline{\Omega})$, hence

$$c = \sup_{u\in E\setminus\{0\}} \frac{\|u\|_\infty}{\|u\|} < +\infty. \tag{2.3}$$

Proposition 2.1. *Let $I : E \to E^*$ be the operator defined by*

$$I(u)\varphi = \int_\Omega \left(|\nabla u|^{p(x)-2}\nabla u \nabla\varphi + |u|^{p(x)-2}u\varphi\right) dx$$

for all $u, \varphi \in E$. Then I admits a continuous inverse on E^*.

Proof.
Denoting by $\langle ., . \rangle$ the usual inner product in $I\!R^N$. We have for some positive constant C_1

$$\frac{\langle I(u), u\rangle}{\|u\|} \geq C_1 \frac{I_{p(x)}(u)}{\|u\|} \geq C_1\|u\|^{p^- -1}.$$

Hence,

$$\lim_{\|u\|\to+\infty} \frac{\langle I(u), u\rangle}{\|u\|} = +\infty.$$

Then I is coercive. Moreover, by simple arguments we can easily verify that I is hemicontinuous. It remains to show that I is uniformly monotone. Indeed, recall first the well known inequality

$$|x-y|^m \leq 2^m \langle |x|^{m-2}x - |y|^{m-2}y, x-y\rangle,\ \forall x, y \in I\!R^N, \forall m \geq 2.$$

Thus, it is easy to see that

$$\begin{aligned}\langle I(u_1) - I(u_2), u_1 - u_2\rangle &\geq \frac{1}{2^{p^+}} \int_\Omega |\nabla u_1 - \nabla u_2|^{p(x)} + |u_1 - u_2|^{p(x)}\, dx \\ &\geq c(p^+)\|u_1 - u_2\|^{p^+},\end{aligned}$$

for every u_1 and u_2 belonging to E. This means that I is uniformly monotone operator in E. Therefore, the conclusion of Proposition 2.1 follows directly from the result (Theorem 26. A) of Zeidler.[19]

Our main result are the following

Theorem 2.1. *Suppose $p^- > N$ and let α satisfying* (1.2). *Assume that $f : \Omega \times I\!R \to I\!R$ is a function which is measurable in Ω and C^1 in $I\!R$ satisfying (1.1), (1.3) and (1.4). Then, there exist an open interval A of $]0, +\infty[$ and a positive real number t such that, for every $\lambda \in A$ and every function $g : \Omega \times I\!R \to I\!R$ which is measurable in Ω and C^1 in $I\!R$ satisfying (1.5), there exists $\sigma > 0$ such that for each $\mu \in [0, \sigma]$ the system (P_1) has at least three weak solutions whose norms in E are less than t.*

3. Proof of the main result

We begin by setting

$$\Phi(u) = \int_\Omega \frac{1}{p(x)} (|\nabla u|^{p(x)} + |u|^{p(x)})\, dx$$

$$J(u) = -\int_\Omega F(x, u)\, dx$$

and

$$\Psi(u) = \int_\Omega \frac{1}{\gamma(x)} |u|^{\gamma(x)}\, dx$$

for each $u \in E$. It is well known that Φ and J are well defined and continuously Gâteaux differentiable with

$$\Phi^{'}(u)\varphi = \int_\Omega (|\nabla u|^{p(x)-2} \nabla u \nabla \varphi + |u|^{p(x)-2} u\varphi)\, dx$$

and

$$J^{'}(u)\varphi = -\int_{\Omega} f(x,u)\varphi\, dx$$

for all u, φ in E. Note that $J^{'}$ is compact and Φ is clearly weakly lower semi-continuous and bounded on each bounded subset of E. Proposition 2.1 ensures that $\Phi^{'}$ admits a continuous inverse on E^{*}. Moreover, it is easy to see that

$$\lim_{\|u\| \longrightarrow +\infty} (\Phi(u) + \lambda J(u)) = +\infty$$

for all $\lambda \in]0, +\infty[$. Indeed, since E is embedded in $C^{0}(\overline{\Omega})$, we have

$$J(u) \geq -|\Omega|(\frac{c_1}{\alpha^{-}}\|u\|_{\infty}^{\alpha^{+}} + c_2\|u\|_{\infty}),$$

where $|\Omega|$ denotes the measure of Ω. Then, the coercivity of $\Phi + \lambda J$ can be easily deduced from (2.1) since $\alpha^{+} < p^{-}$ and $\lambda > 0$ and from the fact that $\|u\|_{\infty} \leq D\|u\|$ for some positive constant D. Then (i) is verified.

In the sequel, we will verify the conditions (ii) and (iii) of Theorem 1.1. By (1.3) and (1.4), it follows that $F(x,t)$ is increasing for $t \in (1, \infty)$ and decreasing for $t \in (0,1)$, uniformly with respect to x and $F(x,0) = 0$ is obvious. Also, by (1.4), we can choose $\delta > 1$ such that

$$F(x,u) \geq 0 = F(x,0) \geq F(x,\tau), \;\; \forall u > \delta,\, \tau \in]0,1[.$$

Let b, d be two numbers such that $0 < b < \min\{1, c\}$, with c given by (2.3) and $d > \delta$ such that $d^{p^{-}}|\Omega| > 1$. Then we obtain

$$\int_{\Omega} \sup_{0 \leq u \leq b} F(x,u) \leq 0 < \frac{1}{c^{p^{+}}}\frac{b^{p^{+}}}{d^{p^{-}}}\int_{\Omega} F(x,d)\, dx.$$

Define the real number r by

$$r = \frac{1}{p^{+}}\left(\frac{b}{c}\right)^{p^{+}}.$$

By choosing

$$u_0(x) = 0 \text{ and } u_1(x) = d \quad \text{for every} \quad x \in \Omega$$

we have

$$\Phi(u_0) = J(u_0) = 0, \Phi(u_1) = \int_{\Omega} \frac{1}{p(x)}|d|^{p(x)}\, dx$$

and

$$J(u_1) = -\int_{\Omega} F(x,d)\, dx.$$

Then we clearly have

$$\Phi(u_1) \geq \frac{1}{p^+} d^{p^-} |\Omega| > r.$$

Thus we deduce that

$$\Phi(u_0) < r < \Phi(u_1).$$

Then (ii) in Theorem 1.1 is verified.
On the other hand, we have

$$-\frac{(\Phi(u_1) - r)J(u_0) + (r - \Phi(u_0))J(u_1)}{\Phi(u_1) - \Phi(u_0)} = r\frac{\displaystyle\int_\Omega F(x,d)\,dx}{\displaystyle\int_\Omega \frac{1}{p(x)}|d|^{p(x)}\,dx} > 0.$$

We also have

$$\frac{1}{p^+} I_{p(x)}(u) \leq \Phi(u),$$

which implies that

$$I_{p(x)}(u) \leq p^+ r < 1,$$

for all $x \in \Omega$ and for all $u \in E$ such that $\Phi(u) \leq r$. Using (2.1), we get $\|u\| \leq 1$. This implies that

$$\frac{1}{p^+}\|u\| \leq \Phi(u) \leq r.$$

Taking into account that

$$|u(x)| \leq c(p^+ r)^{\frac{1}{p^+}} < b$$

for all $x \in \Omega$ and for all $u \in E$ such that $\Phi(u) \leq r$, with $c = \sup_{u\in E} \frac{\|u\|_\infty}{\|u\|}$. It follows

$$-\inf_{u\in\Phi^{-1}(]-\infty,r])} J(u) = \sup_{\Phi(u)\leq r} -J(u) \leq \int_\Omega \sup_{0\leq u\leq b} F(x,u)\,dx \leq 0.$$

Consequently, we obtain

$$\inf_{u\in\Phi^{-1}(]-\infty,r])} J(u) > \frac{(\Phi(u_1) - r)J(u_0) + (r - \Phi(u_0))J(u_1)}{\Phi(u_1) - \Phi(u_0)}.$$

This means that condition (iii) in Theorem 1.1 is verified.

Moreover, since the function $G : \Omega \times I\!R \longrightarrow I\!R$ is a measurable in Ω and C^1 in $I\!R \times I\!R$ satisfying the condition (1.5), then the functional

$\Psi(u) = \int_\Omega \left(\int_0^{u(x)} g(x,t)\,dt \right) dx$ is well defined and continuously Gâteaux differentiable on E, with compact derivative, and one has

$$\Psi^{'}(u)\varphi = \int_\Omega g(x,u(x))\varphi(x)\,dx$$

for all $u, \varphi \in E$. So, in view of Proposition 2.1 and Theorem 1.1, the proof of Theorem 2.1 is achieved.

Corollary 3.1. *Suppose $p^- > N$. Let α satisfying* (1.2). *Then there exists an open interval A of $]0,+\infty[$ and a positive real number t such that, for every $\lambda \in A$ and every γ, with $\gamma > 1$ there exists $\sigma > 0$ such that for each $\mu \in [0,\sigma]$ the system $(P_1)'$ has at least two weak nonzero solutions whose norms in E are less than t.*

Proof.
Let

$$\Phi(u) = \int_\Omega \frac{1}{p(x)}(|\nabla u|^{p(x)} + |u|^{p(x)})\,dx$$

$$J(u) = -\int_\Omega (\frac{1}{\alpha(x)}|u|^{\alpha(x)} - u)\,dx$$

and

$$\Psi(u) = \int_\Omega \frac{1}{\gamma(x)}|u|^{\gamma(x)}\,dx$$

for each $u \in E$. It is well known that Φ and J are well defined and continuously Gâteaux differentiable with

$$\Phi^{'}(u)\varphi = \int_\Omega (|\nabla u|^{p(x)-2}\nabla u \nabla\varphi + |u|^{p(x)-2}u\varphi)\,dx$$

and

$$J^{'}(u)\varphi = -\int_\Omega |u|^{\alpha(x)-2}u\varphi\,dx$$

for all u, φ in E. Note that $J^{'}$ is compact and Φ is clearly weakly lower semi-continuous and bounded on each bounded subset of E. Proposition 2.1 ensures that $\Phi^{'}$ admits a continuous inverse on E^*. Moreover, it is easy to see that

$$\lim_{\|u\| \longrightarrow +\infty} (\Phi(u) + \lambda J(u)) = +\infty$$

for all $\lambda \in]0, +\infty[$. Consider now

$$H(x,u) = \frac{1}{\alpha(x)}|u|^{\alpha(x)} - u.$$

Since $\alpha(x) > 0$ for all $x \in \Omega$ we have

$$\lim_{|u|\to\infty} H(x,u) = +\infty.$$

Choose $\delta > 1$ such that

$$\frac{1}{\alpha(x)}|u|^{\alpha(x)} - u > 0$$

for all $u > \delta$. Hence we get

$$H(x,u) \geq 0 = H(x,0) \geq H(x,\tau), \;\; \forall u > \delta, \, \tau \in]0,1[.$$

Let us choose b, d two real numbers and u_0 and u_1 the same as in the proof of Theorem 2.1. Then we obtain (ii) and (iii) of Theorem 1.1 when adapting the techniques given in the proof of Theorem 2.1. On the other hand, the functional

$$\Psi(u) = \int_\Omega \frac{1}{\gamma(x)}|u|^{\gamma(x)}\, dx$$

is continuously Gâteaux differentiable on E, with compact derivative. So, in view of Theorem 1.1, the proof is completed.

Acknowledgement

The research of S. El Manouni is supported by Al-Imam University project No. 28/12.

References

1. G. Anello and G. Cordaro, Existence of solutions of the Neumann problem involving the p-Laplacian via variational principle of Ricceri. Arch. Math. (Basel)79 (2002)274-287.
2. G. Bonanno, A minimax inequality and its applications to ordinary differential equations. J. Math. Anal. Appli. 270(2002) 210-219.
3. G. Bonanno, Some remarks on a three critical points theorem, Nonlinear Anal. 54(2003) 651-665.
4. G. Bonanno and P. Candito, Three solutions to a Neumann problem for elliptic equations involving the p-Laplacian. Arch. Math. 80(2003) 424-429.
5. G. Dai, Three solutions for a Neumann-type differential inclusion problem involving the p(x)-Laplacian, Nonlinear Anal. to appear
6. S. El Manouni and M. Kbiri Alaoui, A result on elliptic systems with Neumann conditions via Ricceri's three critical points theorem, Nonlinear Anal. to appear

7. O. Kováčik and J. Rákosník, On spaces $L^{p(x)}$ and $W^{1,p(x)}$, Czechoslovak Math. J. 41(1991) 592-618.
8. X. L. Fan, J. Shen and D. Zhao, Sobolev embedding theorems for spaces $W^{k,p(x)}(\Omega)$, J. Math. Anal. Appl. 262(2001) 749-760.
9. X. L. Fan and D. Zhao, On the spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$, J. Math. Anal. Appl. 263(2001) 424-446.
10. M. Mihăilescu, Existence and multiplicity of solutions for a Neumann problem involving the p(x)-Laplace operator, Nonlinear Anal., 67(2007) 1419-1425.
11. J. Musielak, Orlicz spaces and modular spaces. Lecture Notes in Mathematics, 1034. Springer-Verlag, Berlin (1983).
12. B. Ricceri, On three critical points theorem, Arch. Math. (Basel) 75 (2000), 220-226.
13. B. Ricceri, A three critical points theorem revisited. Nonlinear Anal. to appear (2008).
14. B. Ricceri, A general variational principle and some of its applications. J. Comput. Appl. Math. 113(2000) 401-410.
15. M. Ruzicka, Electrorheological fluids: Modelling and Mathematical Theory, Lecture Notes in Math., Springer-Verlag, Berlin (2002).
16. M. Ruzicka, Flow of shear dependent electrorheological fluids. C. R. A. S. Sci. Paris. Sér. I. Math. 329(1999) 393-398.
17. X. Shi and X. Ding, Existence and multiplicity of solutions for a general p(x)-Laplacian Neumann problem, Nonlinear Anal. to appear
18. J. Simon, régularité de la solution d'une equation non linéaire dans $\mathbb{R}^N$. LMN 665, P. Benilan ed., Berlin-Heidelberg-New York 1978.
19. E. Zeidler, Nonlinear functional analysis and its applications. Vol. II/B. Berlin-Heidelberg-New York 1978.

Positive solutions with changing sign energy to nonhomogeneous elliptic problem of fourth order

M. Talbi* and N. Tsouli†

Département de Mathématiques et Informatique Faculté des Sciences
Université Mohamed 1, Oujda, Morocco
*E-mails: *talbi_md@yahoo.fr, †tsouli@hotmail.com*

In this paper, we study the existence for two positive solutions to a nonhomogeneous elliptic equation of fourth order with a parameter λ such that $0 < \lambda < \hat{\lambda}$. The first solution has a negative energy while the energy of the second one is positive for $0 < \lambda < \lambda_0$ and negative for $\lambda_0 < \lambda < \hat{\lambda}$. The values λ_0 and $\hat{\lambda}$ are given under variational form and we show that every corresponding critical point is solution of the nonlinear elliptic problem (with a suitable multiplicative term).

Keywords: Ekeland's principle; p-Laplacian operator; Palais-Smale condition.

1. Introduction

We consider the problem with Navier boundary conditions

$$(P_\lambda) \qquad \begin{cases} \Delta_p^2 u = \lambda |u|^{q-2}u + |u|^{r-2}u & \text{in} \quad \Omega \\ u = \Delta u = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

Here Ω is a smooth domain in $\mathbb{R}^N$ $(N \geq 1)$, Δ_p^2 is the p-biharmonic operator defined by $\Delta_p^2 u = \Delta(| \Delta u |^{p-2} \Delta u)$, λ is a positive parameter, p, q and r are reals such that

$$1 < q < p < r < p_2^*, \text{ where } \begin{cases} p_2^* = \dfrac{Np}{N-2p} & \text{if } p < N/2, \\ p_2^* = +\infty & \text{if } p \geq N/2. \end{cases}$$

Such kind of problems with combined concave and convex nonlinearities were studied recently by several authors[2–7,9–11,17] in the case of operator Δ_p. Our main results here can be summarized as follows:
Let us put $X = W_0^{2,p}(\Omega) \cap W^{2,p}(\Omega)$. We find two characteristic values λ_0

and $\hat{\lambda}$ ($\lambda_0 < \hat{\lambda}$) under variational form, i.e.

$$(V) \qquad \lambda_0 = C_0(p,q,r) \inf_{u \in X\setminus\{0\}} F(u) \quad \text{and} \quad \hat{\lambda} = \hat{C}(p,q,r) \inf_{u \in X\setminus\{0\}} F(u),$$

such that two branches of positive solutions to (P_λ) exist for $\lambda \in]0, \hat{\lambda}[$ (the functional F will be given below). Moreover, the energy of the first positive solution is negative for $\lambda \in]0, \hat{\lambda}[$ while the energy of the second positive solution changes sign at λ_0, i.e. it is positive for$\lambda \in]0, \lambda_0[$ and negative for $\lambda \in]\lambda_0, \hat{\lambda}[$. Notice that these two positive solutions are found simultaneously and that our approach does not use the mountain-pass lemma.

On the other hand, we show that every solution of (V) is a solution of the problem (P_λ) (with a suitable multiplicative term). This second point lets expect that the first nonlinear eigenvalue ζ of (V), i.e.

$$\zeta = sup\{\lambda > 0 : (P_\lambda) \quad \text{has a nonnegative solution}\}$$

may satisfy a variational problem similar to (V) (see[4] for $p = 2$). Let us precise that $\hat{\lambda}$ coincides with ζ when $q \to p$ and that $\hat{\lambda}$ constitutes a good minoration of ζ in the general case $1 < q < p$.

We consider the transformation of Poisson problem used by P.Drábek and M.ôtani (cf.[12]):
We recall some properties of the Dirichlet problem for the Poisson equation:

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1}$$

It is well known that (1) is uniquely solvable in $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ for all $f \in L^p(\Omega)$ and for any $p \in]1, +\infty[$.
We denote by :
$X = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$,
$\|u\|_p = (\int_\Omega |u|^p dx)^{1/p}$ the norm in $L^p(\Omega)$,
$\|u\|_{2,p} = (\|\Delta u\|_p^p + \|u\|_p^p)^{1/p}$ the norm in X,
$\|u\|_\infty$ the norm in $L^\infty(\Omega)$,
and $< .,. >$ is the duality bracket between $L^p(\Omega)$ and $L^{p'}(\Omega)$, where $p' = p/(p-1)$. Denote by Λ the inverse operator of $-\Delta : X \to L^p(\Omega)$.
The following lemma gives us some properties of the operator Λ (cf.,[12 16])

Lemma 1.1.

(i) (Continuity): There exists a constant $c_p > 0$ *such that*

$$\|\Lambda f\|_{2,p} \le c_p \|f\|_p$$

holds for all $p \in]1, +\infty[$ *and* $f \in L^p(\Omega)$.

(ii) *(Continuity) Given* $k \in \mathbb{N}^*$, *there exists a constant* $c_{p,k} > 0$ *such that*

$$\|\Lambda f\|_{W^{k+2,p}} \leq c_{p,k}\|f\|_{W^{k,p}}$$

holds for all $p \in]1, +\infty[$ *and* $f \in W^{k,p}(\Omega)$.

(iii) *(Symmetry) The following identity:*

$$\int_\Omega \Lambda u \cdot v dx = \int_\Omega u \cdot \Lambda v dx$$

holds for all $u \in L^p(\Omega)$ *and* $v \in L^{p'}(\Omega)$ *with* $p \in]1, +\infty[$.

(iv) *(Regularity) Given* $f \in L^\infty(\Omega)$, *we have* $\Lambda f \in C^{1,\alpha}(\bar{\Omega})$ *for all* $\alpha \in]0, 1[$; *moreover, there exists* $c_\alpha > 0$ *such that*

$$\|\Lambda f\|_{C^{1,\alpha}} \leq c_\alpha \|f\|_\infty.$$

(v) *(Regularity and Hopf-type maximum principle) Let* $f \in C(\bar{\Omega})$ *and* $f \geq 0$ *then* $w = \Lambda f \in C^{1,\alpha}(\bar{\Omega})$, *for all* $\alpha \in]0, 1[$ *and* w *satisfies:* $w > 0$ *in* Ω, $\frac{\partial w}{\partial n} < 0$ *on* $\partial\Omega$.

(vi) *(Order preserving property) Given* $f, g \in L^p(\Omega)$ *if* $f \leq g$ *in* Ω, *then* $\Lambda f < \Lambda g$ *in* Ω.

Remark 1.1. $(\forall u \in X)(\forall v \in L^p(\Omega)) \quad v = -\Delta u \Longleftrightarrow u = \Lambda v.$

Let us denote N_p the Nemytskii operator defined by

$$\begin{cases} N_p(v)(x) = |v(x)|^{p-2}v(x) & \text{if } v(x) \neq 0 \\ N_p(v)(x) = 0 & \text{if } v(x) = 0, \end{cases}$$

and we have $\forall v \in L^p(\Omega), \forall w \in L^{p'}(\Omega)$:

$$N_p(v) = w \Longleftrightarrow v = N_{p'}(w).$$

We define the functionals $P, Q, R : L^p(\Omega) \to \mathbb{R}$ as follows:

$$P(v) = \|v\|_p^p, \quad Q(v) = ||\Lambda v||_q^q \quad \text{and} \quad R(v) = ||\Lambda v||_r^r.$$

The operator Λ enables us to transform problem (P_λ) to an other problem which we will study in the space $L^p(\Omega)$.

Definition 1.1. We say that $u \in X \setminus \{o\}$ is a solution of problem (P_λ), if $v = -\Delta u$ is a solution of the following problem

$$(P'_\lambda) \qquad \begin{cases} \text{Find } v \in L^p(\Omega) \setminus \{o\}, \text{ such that} \\ N_p(v) = \lambda\Lambda(N_q(\Lambda v)) + \Lambda(N_r(\Lambda v)) \quad \text{in} \quad L^{p'}(\Omega). \end{cases}$$

For solutions of (P_λ) we understand critical points of the associated Euler-Lagrange functional $E_\lambda \in \mathcal{C}^1(L^p(\Omega))$, given by

$$E_\lambda(v) = \frac{1}{p}P(v) - \lambda\frac{1}{q}Q(v) - \frac{1}{r}R(v).$$

As in (cf.[13,19]), we introduce the modified Euler-Lagrange functional defined on $\mathbb{R} \times L^p(\Omega)$ by $\tilde{E}_\lambda(t,v) = E(tv)$. If v is an arbitrary element of $L^p(\Omega)$, $\partial_t\tilde{E}_\lambda(.,v)$ (resp. $\partial_{tt}\tilde{E}_\lambda(.,v)$)are the first (resp. second) derivative of the real valued function: $t \mapsto \tilde{E}_\lambda(t,v)$.

2. Preliminary results

Since the functional $\tilde{E}_\lambda$ is even in t and that we are interested by the positive solutions, we limit our study for $t > 0$.

Lemma 2.1. *For every $v \in L^p(\Omega) \setminus \{0\}$, There is a unique $\lambda(v) > 0$ such that the real valued function $t \mapsto \partial\tilde{E}_\lambda(t,v)$ has exactly two positive zeros (resp. one positive zero) if $0 < \lambda < \lambda(v)$ (resp. $\lambda = \lambda(v)$). This function has no zero for $\lambda > \lambda(v)$.*

***Proof*:** Let v be an arbitrary element of $L^p(\Omega) \setminus \{0\}$ and let us write

$$\partial_t\tilde{E}_\lambda(t,v) = t^{q-1}\tilde{F}_\lambda(t,v), \quad \text{where } \tilde{F}_\lambda(t,v) = t^{p-q}P(v) - \lambda Q(v) - t^{r-q}R(v).$$

Then

$$\partial_{tt}\tilde{E}_\lambda(t,v) = (q-1)t^{q-2}\tilde{F}_\lambda(t,v) + t^{q-1}\partial_t\tilde{F}_\lambda(t,v),$$

holds true, with

$$\partial_t\tilde{F}_\lambda(t,v) = t^{p-q-1}[(p-q)P(v) - (r-q)t^{r-p}R(v)].$$

It is clair that the real valued function $t \mapsto \tilde{F}_\lambda(t,v)$ is increasing on $]0,t(v)[$, decreasing on $]t(v),+\infty[$ and attains its unique maximum for $t = t(v)$, where

$$t(v) = \left(\frac{p-q}{r-q}\frac{P(v)}{R(v)}\right)^{\frac{1}{r-p}}. \tag{2}$$

Thus, if $\tilde{F}_\lambda(t(v),v) > 0$ (resp. $\tilde{F}_\lambda(t(v),v) = 0$), the function $t \mapsto \tilde{F}_\lambda(t,v)$ has two positive zeros (resp. one positive zero) and has no zero if $\tilde{F}_\lambda(t(v),v) < 0$. On the other hand, a direct computation gives

$$\tilde{F}_\lambda(t(v),v) = \frac{r-p}{p-q}\left(\frac{p-q}{r-q}\frac{P(v)}{R(v)}\right)^{\frac{r-q}{r-p}}R(v) - \lambda Q(v).$$

We deduce that $\tilde{F}_\lambda(t(v), v) > 0$ (resp. $\tilde{F}_\lambda(t(v), v) < 0$) for $\lambda < \lambda(v)$ (resp. $\lambda > \lambda(v)$) and $\tilde{F}_{\lambda(v)}(t(v), v) = 0$, where

$$\lambda(v) = \hat{c}\frac{P^{\frac{r-q}{r-p}}(v)}{Q(v)R^{\frac{p-q}{r-p}}(v)}, \tag{3}$$

with

$$\hat{c} = \frac{r-p}{p-q}(\frac{p-q}{r-q})^{\frac{r-q}{r-p}}.$$

Hence, if $\lambda \in]0, \lambda(v)[$, the real valued function $t \mapsto \partial_t \tilde{E}_\lambda(t, v)$ has two positive zeros, denoted by $t_1(v, \lambda)$ and $t_2(v, \lambda)$, verifying $0 < t_1(v, \lambda) < t(v) < t_2(v, \lambda)$.

Since $\tilde{F}_\lambda(t_1(v, \lambda), v) = \tilde{F}_\lambda(t_2(v, \lambda), v) = 0$, $\partial_t \tilde{F}_\lambda(t, v) > 0$ for $t < t(v)$ and $\partial_t \tilde{F}_\lambda(t, v) < 0$ for $t > t(v)$, it follows that

$$\partial_{tt}\tilde{E}_\lambda(t_1(v, \lambda), v) > 0 \quad \text{and} \quad \partial_{tt}\tilde{E}_\lambda(t_2(v, \lambda), v) < 0. \tag{4}$$

This means that the real valued function $t \mapsto \tilde{E}_\lambda(t, v)$, $(t > 0)$ achieves its unique local minimum at $t = t_1(v, \lambda)$ and its global maximum at $t = t_2(v, \lambda)$.□

Lemma 2.2. *If we put* $\hat{\lambda} = \inf_{v \in L^p(\Omega)\setminus\{0\}} \lambda(v)$, *then* $\hat{\lambda} > 0$.

***Proof*:** By Sobolev injection theorem, we have $X \hookrightarrow \text{Ł}^q(\Omega)$ and $X \hookrightarrow \text{Ł}^r(\Omega)$. Thus there exists two positive constants c_1 and c_2 such that

$$||\Lambda v||_q \leq c_1||v||_p \quad \text{et} \quad ||\Lambda v||_r \leq c_2||v||_p.$$

Then (3) implies for every $v \in L^p(\Omega) \setminus \{0\}$

$$\lambda(v) \geq \frac{\hat{c}}{c_1^q c_2^{\frac{r(p-q)}{r-p}}} > 0.$$

□

Consider $\lambda \in]0, \hat{\lambda}[$ and let (v_n) be minimizing sequence of $v \mapsto \tilde{E}_\lambda(t_1(v, \lambda), v)$ in $L^p(\Omega) \setminus \{0\}$ (resp. of $v \mapsto \tilde{E}_\lambda(t_2(v, \lambda), v)$).
Put $V_n = t_1(v_n, \lambda)v_n$ and $W_n = t_2(v_n, \lambda)v_n$.

Lemma 2.3. *The sequences* (V_n) *and* (W_n) *verify :*

(i) $\limsup_{n\to+\infty} ||V_n||_p < +\infty$ *(resp.* $\limsup_{n\to+\infty} ||W_n||_p < +\infty$ *)*

(ii) $\liminf_{n\to+\infty} ||V_n||_p > 0$ *(resp.* $\liminf_{n\to+\infty} ||W_n||_p > 0$*)*

***Proof*:** (i) We know that $\partial_t \tilde{E}_\lambda[t_1(v_n,\lambda),v_n)=0$.
Hence

$$||V_n||_p^p=\lambda||\Lambda V_n||_q^q+||\Lambda V_n||_r^r. \tag{5}$$

Suppose that there is a subsequence of (V_n), still denoted by (V_n) such that $\lim_{n\to+\infty}||V_n||_p=+\infty$. Us $r>q$, there exist a constant $c>0$ such that $||\Lambda V_n||_q\leq c||\Lambda V_n||_r$. Then the relation (5) implies that $\lim_{n\to+\infty}||\Lambda V_n||_r=+\infty$. The fact that $0<q<r$ enables us to deduce: $||\Lambda V_n||_q^q=o_n(||\Lambda V_n||_r^r)$.
Then

$$||V_n||_p^p=||\Lambda V_n||_r^r(1+o_n(1)),$$

and

$$E_\lambda(V_n)=||\Lambda V_n||_r^r(\frac{1}{p}-\frac{1}{r}+o_n(1)).$$

which implies that $E_\lambda(V_n)$ tends to $+\infty$ as n goes to $+\infty$ and this is impossible.
The same arguments with a minimizing sequence (v_n) of $v\mapsto\tilde{E}_\lambda(t_2(v,\lambda),v)$ show that $\limsup_{n\to+\infty}||W_n||_p<+\infty$.
(ii) Relation (5) and the fact that $\partial_{tt}\tilde{E}_\lambda[t_1(v_n,\lambda),v_n)>0$, implies

$$(p-1)||V_n||_p^p-\lambda(q-1)||\Lambda V_n||_q^q-(r-1)||\Lambda V_n||_r^r>0. \tag{6}$$

If we combine (5) and (6), we obtain for every $n\in\mathbb{N}$

$$\lambda(p-q)||\Lambda V_n||_q^q+(p-r)||\Lambda V_n||_r^r>0.$$

So

$$\begin{aligned}E_\lambda(V_n)&=\lambda\frac{q-p}{pq}Q(V_n)+\frac{r-p}{pr}R(V_n)\\&\leq\frac{-1}{pq}(\lambda(p-q)Q(V_n)+(p-r)R(v_n))\\&<0.\end{aligned}$$

Suppose that there is a subsequence of (V_n), still denoted by (V_n) such that $\lim_{n\to+\infty}||V_n||_p=0$. By Sobolev injection theorem we deduce that $\lim_{n\to+\infty}||\Lambda V_n||_q=0$ and $\lim_{n\to+\infty}||\Lambda V_n||_r=0$. It follows that $\lim_{n\to+\infty}E_\lambda(V_n)=0$, $i.e$ $\inf_{v\in L^p(\Omega)\setminus\{0\}}\tilde{E}_\lambda(t_1(v,\lambda),v)=0$, which is impossible since $\tilde{E}_\lambda(t_1(v_n,\lambda),v_n)<0$ for every n.

Let (v_n) be a minimizing sequence of $v \mapsto \tilde{E}_\lambda(t_2(v), v)$ in $L^p(\Omega) \setminus \{0\}$. Sinse $\partial_t \tilde{E}_\lambda(t_2(v_n), v_n) = 0$ and $\partial_{tt} \tilde{E}_\lambda(t_2(v_n), v_n) < 0$, it follows that

$$\begin{cases} ||W_n||_p^p - \lambda ||\Lambda W_n||_q^q - ||\Lambda W_n||_r^r = 0, \\ (p-1)\ ||W_n||_p^p - \lambda(q-1)||\Lambda W_n||_q^q - (r-1)||\Lambda W_n||_r^r < 0. \end{cases}$$

Combining the two last inequalities and by Sobolev injection theorem there exist a constant c' such that for every n we have

$$(p-q)||W_n||_p^p < (r-q)||\Lambda W_n||_r^r \leq c'||W_n||_p^r.$$

Hence

$$(p-q) \leq c'||W_n||_p^{r-p}.$$

Now, suppose that there is a subsequence of (W_n), still denoted by (W_n) such that $\lim\limits_{n\to+\infty} ||W_n||_p = 0$. This implies that $p-q \leq 0$. which is impossible since $p > q$. □

Lemma 2.4. *The functionals* $v \mapsto \tilde{E}_\lambda(t_1(v,\lambda), v)$ *and* $v \mapsto \tilde{E}_\lambda(t_2(v,\lambda), v)$ *are bonded below in* $L^p(\Omega)$.

***Proof*:** Let (v_n) be a minimizing sequence of the functional $v \mapsto \tilde{E}_\lambda(t_1(v,\lambda), v)$.
We know that $\partial_t \tilde{E}_\lambda(t_1(v_n,\lambda), v_n) = 0$, then

$$[t_1(v_n,\lambda)]^p||v_n||_p^p = \lambda[t_1(v_n,\lambda)]^q||\Lambda v_n||_q^q + [t_1(v_n,\lambda)]^r||\Lambda v_n||_r^r.$$

Hence

$$\tilde{E}_\lambda(t_1(v_n,\lambda), v_n) = \lambda(\frac{1}{p} - \frac{1}{q})[t_1(v_n,\lambda)]^q||\Lambda v_n||_q^q + (\frac{1}{p} - \frac{1}{r})[t_1(v_n,\lambda)]^r||\Lambda v_n||_r^r.$$

As $p < r$, we conclude that

$$\tilde{E}_\lambda(t_1(v_n,\lambda), v_n) \geq \lambda(\frac{1}{p} - \frac{1}{q})[t_1(v_n,\lambda)]^q||\Lambda v_n||_q^q. \tag{7}$$

Sobolev injection of X in $L^q(\Omega)$ and the fact that $\limsup\limits_{n\to+\infty} ||V_n||_p < +\infty$, implies that there exists c and k positive such that for every n in $\mathbb{N}$, we have $||V_n||_p < k$. and $||\Lambda V_n||_q \leq c||V_n||_p < kc$. As $q < p$, the inequality (7) implies

$$\tilde{E}_\lambda(t_1(v_n,\lambda), v_n) > (\frac{1}{p} - \frac{1}{q})\lambda k^q c^q.$$

We show by the same method that the functional $v \mapsto \tilde{E}_\lambda(t_2(v,\lambda),v)$ is bonded bellow. □

Put

$$\alpha_1(\lambda) = \inf_{v \in L^p(\Omega)\setminus\{0\}} \tilde{E}_\lambda(t_1(v,\lambda),v). \tag{8}$$

$$\alpha_2(\lambda) = \inf_{v \in L^p(\Omega)\setminus\{0\}} \tilde{E}_\lambda(t_2(v,\lambda),v). \tag{9}$$

We have the following lemma:

Lemma 2.5. *If $\lambda \in]0,\hat{\lambda}[$, then*

$$\alpha_1(\lambda) = \inf_{v \in S, v \geq 0} \tilde{E}_\lambda(t_1(v,\lambda),v) \quad \textit{and} \quad \alpha_2(\lambda) = \inf_{v \in S, v \geq 0} \tilde{E}_\lambda(t_2(v,\lambda),v),$$

where S is the unit sphere of $L^p(\Omega)$.

***Proof*:** Let $t > 0$. If $\partial_t \tilde{E}_\lambda(t,v) > 0$, then $t \in]t_1(v,\lambda), t_2(v,\lambda)[$.
Since $|\Lambda v| \leq \Lambda|v|$, we deduce that

$$\partial_t \tilde{E}_\lambda(t_i(|v|,\lambda),v) \geq \partial_t \tilde{E}_\lambda(t_i(|v|,\lambda),|v|) = 0, \quad i = 1,\ 2.$$

It follows that $]t_1(|v|,\lambda), t_2(|v|,\lambda)[\subseteq]t_1(v,\lambda), t_2(v,\lambda)[$.

Hence, $t_1(|v|,\lambda) \geq t_1(v,\lambda)$.

Using the fact that $t \mapsto \tilde{E}_\lambda(t,|v|)$ is decreasing on $]0, t_1(|v|,\lambda)]$, we get

$$\tilde{E}_\lambda(t_1((v,\lambda),|v|) \geq \tilde{E}_\lambda(t_1(|v|,\lambda),|v|)$$

and since $|\Lambda v| \leq \Lambda|v|$, we get

$$\tilde{E}_\lambda(t_1(v,\lambda),v) \geq \tilde{E}_\lambda(t_1(v,\lambda),|v|).$$

Hence we conclude that

$$\tilde{E}_\lambda(t_1(|v|,\lambda),|v|) \leq \tilde{E}_\lambda(t_1(v,\lambda),v).$$

Since $|\Lambda v| \leq \Lambda|v|$ and the function $t \mapsto \tilde{E}_\lambda(t,v)$ is creasing on $[t_1(v,\lambda), t_2(v,\lambda)]$, we obtain

$$\begin{aligned} \tilde{E}_\lambda(t_2(|v|,\lambda),|v|) &\leq \tilde{E}_\lambda(t_2(|v|,\lambda),v) \\ &\leq \tilde{E}_\lambda(t_2(v,\lambda),v). \end{aligned}$$

Finally, we have showed that for every $v \in L^p(\Omega) \setminus \{0\}$

$$\tilde{E}_\lambda(t_i(|v|,\lambda),|v|) \leq \tilde{E}_\lambda(t_i(v,\lambda),v), \quad \text{where} \quad i = 1,\ 2. \tag{10}$$

Moreover, for every $\gamma > 0$, we get

$$\tilde{E}_\lambda(\gamma t, \frac{v}{\gamma}) = \tilde{E}_\lambda(t, v),$$

$$\partial_t \tilde{E}_\lambda(\gamma t, \frac{v}{\gamma}) = \frac{1}{\gamma}\partial_t \tilde{E}_\lambda(t, v),$$

$$\partial_{tt} \tilde{E}_\lambda(\gamma t, \frac{v}{\gamma}) = \frac{1}{\gamma^2}\partial_{tt} \tilde{E}_\lambda(t, v).$$

It follows that

$$t_1(v, \lambda) = \frac{1}{\gamma} t_1(\frac{v}{\gamma}, \lambda), \tag{11}$$

$$t_2(v, \lambda) = \frac{1}{\gamma} t_2(\frac{v}{\gamma}, \lambda). \tag{12}$$

By the virtue of (10), (11) and (12), we conclude that

$$\alpha_1(\lambda) = \inf_{v \in S, v \geq 0} \tilde{E}_\lambda(t_1(v, \lambda), v), \tag{13}$$

$$\alpha_2(\lambda) = \inf_{v \in S, v \geq 0} \tilde{E}_\lambda(t_2(v, \lambda), v), \tag{14}$$

where S is the unit sphere of $L^p(\Omega)$. □

Lemma 2.6. *Let $(v_n) \subset S$ be a minimizing sequence of* (13) *(resp. of* (14)*). Then, $(V_n) := (t_1(v_n, \lambda)v_n)$ (resp. $(W_n) := (t_2(v_n, \lambda)v_n)$) are Palais-Smale sequences for the functional E_λ.*

***Proof*:** We will show this lemma only for the sequence (V_n), the proof for (W_n) can be done in the same way.

Let $\lambda \in]0, \hat{\lambda}[$. Then $\lim_{n \to +\infty} E_\lambda(V_n) = \alpha_1(\lambda)$.

Now we show that $\lim_{n \to +\infty} E'_\lambda(V_n) = 0$.

Notice that for every $v \in L^p(\Omega) \setminus \{0\}$, we have $\partial_t \tilde{E}_\lambda(t_1(v, \lambda), v) = 0$ and $\partial_{tt} \tilde{E}_\lambda(t_1(v, \lambda), v) \neq 0$. The implicit function theorem implies that the functional $v \mapsto t_1(v, \lambda)$ is C^1 since $\tilde{E}_\lambda$ is. Let us introduce the C^1 functional $f_{1,\lambda}$ defined on S by

$$f_{1,\lambda}(v) = \tilde{E}_\lambda(t_1(v, \lambda), v) = E_\lambda(t_1(v, \lambda)v).$$

Hence

$$\alpha_1(\lambda) = \inf_{v \in S} f_{1,\lambda}(v) = \inf_{v \in S, v \geq 0} f_{1,\lambda}(v) \quad \text{and} \quad \lim_{n \to +\infty} f_{1,\lambda}(v_n) = \alpha_1(\lambda).$$

Using the Ekeland variational principle on the complete manifold $(S, || \ ||_p)$ to the functional $f_{1,\lambda}$, we conclude that

$$|f'_{1,\lambda}(v_n)(\varphi)| \leq \frac{1}{n}||\varphi||_p, \quad \text{for every} \quad \varphi \in T_{v_n}S,$$

where $T_{v_n}S$ is the tangent space to S at the point v_n.
Moreoever, since $\partial_t \tilde{E}_\lambda(t_1(v_n,\lambda), v_n) \equiv 0$, then for every $\varphi \in T_{v_n}S$, one has

$$\begin{aligned} f'_{1,\lambda}(v_n)(\varphi) &= \partial_t \tilde{E}_\lambda(t_1(v_n,\lambda), v_n)\partial_v t_1(v_n,\lambda)(\varphi) \\ &\quad + \partial_v \tilde{E}_\lambda(t_1(v_n,\lambda), v_n)(\varphi) \\ &= \partial_v \tilde{E}_\lambda(t_1(v_n,\lambda), v_n)(\varphi), \end{aligned}$$

where $\partial_v t_1(v_n,\lambda)$ denotes the derivative of $t_1(.,\lambda)$ with respect to its first variable at the point (v_n,λ).

Furthermore, let

$$\begin{aligned} P : L^p(\Omega)\backslash\{0\} &\to \mathbb{R} \times S \\ v &\mapsto (P_1(v), P_2(v)) = (\| \ v \ \|_p, \frac{v}{\| \ v \ \|_p}). \end{aligned}$$

Applying Hölder's inequality, we get for every $(v, \ \varphi) \in L^p(\Omega)\backslash\{0\} \times L^p(\Omega)$:

$$\|P'_2(v)(\varphi)\|_p \leq 2\frac{\|\varphi\|_p}{\|v\|_p}.$$

From lemma 2.3 and by the fact that $\|V_n\|_p = t(v_n,\lambda)$, there exists positive constant C such that

$$t_1(v_n,\lambda) \geq C, \forall n \in \mathbb{N}.$$

Hence for every $\varphi \in L^p(\Omega)$, we obtain

$$\begin{aligned} |E'(V_n)(\varphi)| &= |\partial_t \tilde{E}_\lambda(P_1(V_n), P_2(V_n))P'_1(V_n)(\varphi) \\ &\quad + \partial_v \tilde{E}_\lambda(P_1(V_n), P_2(V_n))P'_2(V_n)(\varphi)| \\ &= |\partial_v \tilde{E}_\lambda(t(v_n), v_n)P'_2(V_n)(\varphi)| \\ &= |f'_{1,\lambda}(v_n)P'_2(V_n)(\varphi)| \\ &\leq \frac{1}{n} \| \ P'_2(V_n)(\varphi) \ \|_p \\ &\leq \frac{2}{n}\frac{\| \ \varphi \ \|_p}{C} \end{aligned}$$

We easily conclude that

$$\lim_{n\to+\infty} E'(V_n) = 0 \quad \text{in} \quad L^{p'}(\Omega).$$

□

Remark 2.1. Until now, the minimizing sequences we consider are in S and are nonnegative.

3. Existence results

Theorem 3.1. *Let $1 < q < p < r < p_2^*$ and $\lambda \in \,]0,\ \hat{\lambda}[$. Then the problem (P_λ) has at least two positive solutions.*

***Proof*:** We will use the notations of the previous lemmas.
Since the sequences (V_n) and (W_n) are Palais-Smale for the functional E_λ, it suffices to show that E_λ $(0 < \lambda < \hat{\lambda})$ satisfy Palais-Smale condition.

By lemma 2.3, we deduce that (V_n) is bonded in $L^p(\Omega)$. Passing if necessary to a subsequence, we get

$$\begin{cases} V_n \rightharpoonup V_1 \quad \text{in} \quad L^p(\Omega), \\ \Lambda V_n \rightharpoonup \Lambda V_1 \quad \text{in} \quad X, \\ \Lambda V_n \to \Lambda V_1 \quad \text{in} \quad L^r(\Omega), \quad (\text{and in} \quad L^q(\Omega)). \end{cases} \tag{15}$$

On the other hand we have,

$$\langle N_p(V_n), V_n - V_1 \rangle = \langle\ E'_\lambda(V_n), V_n - V_1 \rangle + \lambda \int_\Omega N_q(\Lambda V_n)(\Lambda V_n - \Lambda V_1)dx$$
$$+ \int_\Omega N_r(\Lambda V_n)(\Lambda V_n - \Lambda V)dx.$$

Moreover, $E'_\lambda(V_n) \to 0$, $N_q(\Lambda V_n) \to N_q(\Lambda V_1)$ and $N_r(\Lambda V_n) \to N_r(\Lambda V_1)$.

Then $\langle N_p(V_n), V_n - V_1 \rangle \to 0$.

The fact that N_p is $(S+)$ type implies that $V_n \to V_1$ in $L^p(\Omega)$.

We know that for any minimizing sequence (v_n) of (13), there is a subsequence still denoted by (v_n) such that $V_n = t_1(v_n, \lambda)v_n$ and $t_1(v_n, \lambda) = ||V_n||_p$. Hence

$$t_1(v_n, \lambda) \to ||V_1||_p = t_1,$$

which implies that

$$v_n \to V_1/t_1 = v_1, \quad \text{and} \quad t_1 = t_1(v_1, \lambda),$$

where $v_1 \in S$.

In the same way, for any minimizing sequence $(v_n) \subset S$ of (14), passing if necessary to a subsequence, there is $t_2 \in \,]0, +\infty[$ such that

$$\begin{cases} t_2(v_n, \lambda)v_n \to t_2 \quad \text{in} \quad \mathbb{R}, \\ \qquad\qquad v_n \to v_2 = V_2/t_2, \end{cases}$$

where V_2 is the limit of the sequence $(W_n) := (t_2(v_n, \lambda)v_n)$ in $L^p(\Omega)$ and $t_2 = ||V_2||_p = t_2(v_2, \lambda)$.

At this stage, it is easy to see that $V_1 \neq V_2$. Indeed, since $\partial_{tt}\tilde{E}_\lambda(t_1(v_1,\lambda),v_1) > 0$ and $\partial_{tt}\tilde{E}_\lambda(t_2(v_2,\lambda),v_2) < 0$, it follows that $\partial_{tt}E_\lambda(t_1,V_1/t_1) > 0$ and $\partial_{tt}E_\lambda(t_2,V_2/t_2) < 0$. This achieves the proof. $\square$

In the sequel the solutions V_1 and V_2 of (P'_λ), for $\lambda \in]0,\hat{\lambda}[$, will be denoted by $V_{1,\lambda}$ and $V_{2,\lambda}$. Also, $t_{1,\lambda}$, $t_{2,\lambda}$, $v_{1,\lambda}$ and $v_{2,\lambda}$ will stand for $t_1(v_1,\lambda)$, $t_2(v_2,\lambda)$, v_1 and v_2 respectively.

Theorem 3.2. *Let $1 < q < p < r < p_2^*$. Then*

$$(i) \quad E_\lambda(V_{1,\lambda}) < 0 \quad \textit{for} \quad \lambda \in\ ]0,\hat{\lambda}[,$$

$$(ii) \begin{cases} E_\lambda(V_{2,\lambda}) > 0 & \textit{for } \lambda \in\ ]0,\lambda_0[, \\ E_\lambda(V_{2,\lambda}) < 0 & \textit{for } \lambda \in\ ]\lambda_0,\hat{\lambda}[, \end{cases}$$

where

$$\lambda_0 = \frac{q}{r}\left(\frac{r}{p}\right)^{\frac{r-q}{r-p}}\hat{\lambda}.$$

***Proof*:** (i) Let us recall that $\partial_t\tilde{E}_\lambda(t_{1,\lambda},v_{1,\lambda}) = 0$ and $\partial_{tt}\tilde{E}_\lambda(t_{1,\lambda},v_{1,\lambda}) > 0$. Then

$$\begin{cases} P(V_{1,\lambda}) - \lambda Q(V_{1,\lambda}) - R(V_{1,\lambda}) = 0, \\ (p-1)P(V_{1,\lambda}) - \lambda(q-1)Q(V_{1,\lambda}) - (r-1)R(V_{1,\lambda}) > 0. \end{cases}$$

Using the fact that $1 < q < p < r$, we get

$$\lambda(p-q)Q(V_{1,\lambda}) + (p-r)R(V_{1,\lambda}) > 0.$$

Hence

$$\begin{aligned} E_\lambda(V_{1,\lambda}) &= \lambda\frac{q-p}{pq}Q(V_{1,\lambda}) + \frac{r-p}{pr}R(V_{1,\lambda}) \\ &\leq \frac{-1}{pq}(\lambda(p-q)Q(V_{1,\lambda}) + (p-r)R(v_{1,\lambda})) \\ &< 0. \end{aligned}$$

(ii) Let v be an arbitrary element of $L^p(\Omega)\setminus\{0\}$ and let us write

$$\tilde{E}_\lambda(t,v) = t^q\tilde{G}_\lambda(t,v), \quad \text{where} \quad \tilde{G}_\lambda(t,v) = \frac{t^{p-q}}{p}P(v) - \frac{\lambda}{q}Q(v) - \frac{t^{r-q}}{r}R(v).$$

It follows that

$$\partial_t\tilde{E}_\lambda(t,v) = qt^{q-1}\tilde{G}_\lambda(t,v) + t^q\partial\tilde{G}_\lambda(t,v),$$

with

$$\partial_t \tilde{G}_\lambda(t,v) = t^{p-q-1}(\frac{p-q}{p}P(v) - \frac{r-q}{r}t^{r-p}R(v)).$$

It is clear that the real valued function $t \to \tilde{G}_\lambda(t,v)$ is increasing on $]0, t_0(v)[$, decreasing on $]t_0(v), +\infty[$ and attains its unique maximum for $t = t_0(v)$, where

$$t_0(v) = (\frac{r}{p})^{\frac{1}{r-p}} t(v), \tag{16}$$

and $t(v)$ is defined by the relation (2).

On the other hand, a direct computation gives

$$\tilde{G}_\lambda(t_0(v), v) = \frac{1}{r}(\frac{r}{p})^{\frac{r-q}{r-p}} \frac{r-p}{p-q}(\frac{p-q}{r-q}\frac{P(v)}{R(v)})^{\frac{r-q}{r-p}} R(v) - \lambda\frac{Q(v)}{q}.$$

Similarly, $\tilde{G}_\lambda(t_0(v), v) > 0$ (resp. $\tilde{G}_\lambda(t_0(v), v) < 0$) if $\lambda < \lambda_0(v)$ (resp. $\lambda > \lambda_0(v)$) and
$\tilde{G}_{\lambda_0(v)}(t_0(v), v) = 0$, where

$$\lambda_0(v) = \frac{q}{r}(\frac{r}{p})^{\frac{r-q}{r-p}} \lambda(v), \tag{17}$$

with $\lambda(v)$ given by (3). Thus, we get

$$\begin{cases} \tilde{E}_\lambda(t_0(v), v) > 0 & \text{if} \quad \lambda < \lambda_0(v), \\ \tilde{E}_\lambda(t_0(v), v) = 0 & \text{if} \quad \lambda = \lambda_0(v), \\ \tilde{E}_\lambda(t_0(v), v) < 0 & \text{if} \quad \lambda > \lambda_0(v). \end{cases} \tag{18}$$

First, since the function

$$\begin{aligned}]0,1[&\to \mathbb{R} \\ t &\to \frac{\ln t}{1-t} \end{aligned}$$

is increasing, then for every real numbers x and y such that $0 < x < y$, one has

$$\ln(\frac{1}{x}) > \frac{1-x}{1-y}\ln(\frac{1}{y}) = \ln(\frac{1}{y})^{\frac{1-x}{1-y}},$$

and consequently

$$0 < x(1/y)^{\frac{1-x}{1-y}} < 1.$$

In the particular case $x = \frac{q}{r}$ and $y = \frac{p}{r}$, we get

$$0 < \frac{q}{r}(\frac{r}{p})^{\frac{r-q}{r-p}} < 1,$$

and therfore $0 < \lambda_0(v) < \lambda(v)$.

Moreover, for every $v \in L^p(\Omega) \setminus \{0\}$, one has $\tilde{G}_{\lambda_0(v)}(t,v) < 0$ for $t \in]0,+\infty[\setminus\{t_0(v)\}$ and $\tilde{G}_{\lambda_0(v)}(t_0(v),v) = 0$. Hence, the real valued function $t \to \tilde{E}_{\lambda_0(v)}(t,v), (t > 0)$, attains its unique maximum at $t = t_0(v)$ and we obtain the following interesting identity

$$t_2(v, \lambda_0(v)) = t_0(v). \tag{19}$$

On the other hand, let

$$\lambda_0 = \inf_{v \in L^p(\Omega)\setminus\{0\}} \lambda_0(v). \tag{20}$$

(3) and (16) implies that

$$\lambda_0(v) = \frac{p}{q}(\frac{r}{p})^{\frac{r-q}{r-p}} \hat{c} \frac{P^{\frac{r-q}{r-p}}(v)}{Q(v)R^{\frac{p-q}{r-p}}(v)}.$$

Let us put

$$M = \{v \in L^p(\Omega), Q(v)R^{\frac{p-q}{r-p}}(v) = 1\}.$$

It is clair that M is weakly closed.

Moreover, the functional $v \mapsto P^{\frac{r-q}{r-p}}(v)$ is weakly lower semi-continuous and coercive on M. Thus this functional attaints its minimum on M. The homogeneities of $v \mapsto P^{\frac{r-q}{r-p}}(v)$ and $v \mapsto Q(v)R^{\frac{p-q}{r-p}}(v)$ enables us to conclude that there is $v^* \in S$ such that

$$\inf_{v \in M} \lambda_0(v) = \inf_{v \in L^p(\Omega)\setminus\{0\}} \lambda_0(v) = \inf_{v \in S} \lambda_0(v) = \lambda_0(v^*) = \lambda_0.$$

Now, let $\lambda \in]0, \lambda_0[$, Then, for every $v \in L^p(\Omega) \setminus \{0\}$ one has $\lambda < \lambda_0(v)$ and consequently, $\tilde{E}_\lambda(t_0(v), v) > 0$ holds from (18). Then the function $t \mapsto \tilde{E}_\lambda(t,v)$, $(t > 0)$ attains its maximum at $t_2(v,\lambda)$ such that $\tilde{E}_\lambda(t_2(v,\lambda), v) > 0$ for every $v \in L^p(\Omega) \setminus \{0\}$. In particular, we have $\tilde{E}_\lambda(t_2(v_{2,\lambda}, \lambda), v_{2,\lambda}) > 0$, *i.e.* $E_\lambda(V_{2,\lambda}) > 0$.

If $\lambda = \lambda_0$, then

$$\begin{aligned} E_{\lambda_0}(V_{2,\lambda_0}) &= \tilde{E}_{\lambda_0}(t_2(v_{2,\lambda_0}), v_{2,\lambda_0}) \\ &= \inf_{v \in S} \tilde{E}_{\lambda_0}(t_2(v, \lambda_0), v) \\ &\leq \tilde{E}_{\lambda_0}(t_2(v^*, \lambda_0(v^*)), v^*) \\ &= \tilde{E}_{\lambda_0(v^*)}(t_0(v^*), v^*) \\ &= 0, \end{aligned}$$

which implies that $E_{\lambda_0}(V_{2,\lambda_0}) \leq 0$. In addition, it is known from (18) that $\tilde{E}_{\lambda_0}(t_0(v), v) \geq 0$, for every $v \in L^p(\Omega) \setminus \{0\}$. Then, since $\tilde{E}_{\lambda_0}(t_2(v_{2,\lambda_0},\lambda_0),v_{2,\lambda_0})$ is a global maximum of the function $t \mapsto \tilde{E}_{\lambda_0}(t,v_{2,\lambda_0}),\ (t>0)$, we have

$$\tilde{E}_{\lambda_0}(t_2(v_{2,\lambda_0},\lambda_0),v_{2,\lambda_0}) \geq \tilde{E}_{\lambda_0}(t_0(v_{2,\lambda_0}),v_{2,\lambda_0}) \geq 0.$$

We conclude that

$$E_{\lambda_0}(V_{2,\lambda_0}) = \tilde{E}_{\lambda_0}(t_2(v_{2,\lambda_0},\lambda_0),v_{2,\lambda_0}) = 0.$$

Finally, suppose that $\lambda_0 < \lambda < \hat{\lambda}$.

We know that for every $(t,v) \in]0,+\infty[\times L^p(\Omega)\setminus\{0\}$, the real valued function $\lambda \mapsto \tilde{E}_\lambda(t,v)$ is decreasing on $[\lambda_0,\hat{\lambda}]$, hence we deduce

$$\begin{aligned}\tilde{E}_\lambda(t_2(v_{2,\lambda},\lambda),v_{2,\lambda}) &= \inf_{v\in S} \tilde{E}_\lambda(t_2(v,\lambda),v)\\ &\leq \tilde{E}_\lambda(t_2(v^*,\lambda),v^*)\\ &< \tilde{E}_{\lambda_0}(t_2(v,\lambda),v).\end{aligned}$$

Moreover, the real valued function $t \mapsto \tilde{E}_{\lambda_0}(t,v^*),\ (t>0)$, attains its unique maximum for $t=t_0(v^*)$.Then

$$\begin{aligned}\tilde{E}_{\lambda_0}(t_2(v^*,\lambda),v^*) &\leq \tilde{E}_{\lambda_0}(t_0(v^*),v^*)\\ &= \tilde{E}_{\lambda_0(v^*)}(t_0(v^*),v^*)\\ &= 0.\end{aligned}$$

Hence $\tilde{E}_\lambda(t_2(v_{2,\lambda},\lambda),v_{2,\lambda}) < 0$, which achieves this proof. □

Theorem 3.3. *if v^* is a solution of* (20)*, then $t_0(v^*)v^*$ is a solution of* (P'_{λ_0}).

***Proof*:** Let v^* be a solution of (20), then $\lambda_0 = \lambda_0(v^*)$ and for every $h \in L^p(\Omega)$, we have

$$\begin{aligned}E'_{\lambda_0}(t_0(v^*)v^*)(h) &= \frac{1}{p}t_0^{p-1}(v)\langle P'(v),h\rangle - \frac{\lambda_0}{q}t_0^{q-1}(v)\langle Q'(v),h\rangle\\ &\quad -\frac{1}{r}t_0^{r-1}(v)\langle R'(v),h\rangle\\ &= \frac{P(v)(t_0(v))^{p-1}}{p}\Big(\frac{\langle P'(v),h\rangle}{P(v)}\\ &\quad -\frac{p\lambda_0}{q}t_0^{q-p}\frac{\langle Q'(v),h\rangle}{P(v)} - \frac{p}{r}t_0^{r-p}\frac{\langle R'(v),h\rangle}{P(v)}\Big).\end{aligned}$$

By the virtue of relations (2), (3), (16) and (17), a direct computation gives for every $h \in L^p(\Omega)$

$$\frac{p\lambda_0}{q}t_0^{q-p}\frac{\langle Q'(v^*),h\rangle}{P(v^*)} = \frac{r-p}{r-q}\frac{\langle Q'(v),h\rangle}{Q(v^*)},$$

and

$$\frac{p}{r} t_0^{r-p} \frac{\langle R'(v^*), h\rangle}{P(v^*)}) = \frac{p-q}{r-q} \frac{\langle R'(v), h\rangle}{R(v^*)}.$$

Then

$$E'_{\lambda_0}(t_0(v^*)v^*)(h) = K(\frac{r-q}{r-p} \frac{\langle P'(v^*), h\rangle}{P(v^*)} - \frac{\langle Q'(v^*), h\rangle}{Q(v^*)} - \frac{p-q}{r-p} \frac{\langle R'(v^*), h\rangle}{R(v^*)}),$$

where

$$K = \frac{r-p}{r-q} \frac{P(v^*)}{p} [t_0(v^*)]^{p-1}.$$

On the other hand, the relations (3) and (17) implies that for every $h \in L^p(\Omega)$

$$\langle \lambda'_0(v^*), h\rangle = \lambda_0(v^*)(\frac{r-q}{r-p} \frac{\langle P'(v^*), h\rangle}{P(v^*)} - \frac{\langle Q'(v^*), h\rangle}{Q(v^*)} - \frac{p-q}{r-p} \frac{\langle R'(v^*), h\rangle}{R(v^*)}).$$

Since $\langle \lambda'_0(v^*), h\rangle = 0$ for every $h \in L^p(\Omega)$, we deduce that

$$\langle E'_{\lambda_0}(t_0(v^*)v^*), h\rangle = \frac{K}{\lambda_0} \langle \lambda'_0(v^*), h\rangle = 0,$$

for every $h \in L^p(\Omega)$.
Which implies that $t_0(v^*)v^*$ is a solution of (P'_{λ_0}).

Remark 3.1. It is very interesting to notice that in the case of homogeneous Dirichlet boundary condition, we have

$$\lim_{q \to p} \hat{\lambda} = \inf_{v \in L^p(\Omega) \setminus \{0\}} \frac{\int_\Omega |v(x)|^p dx}{\int_\Omega |\Lambda v(x)|^p dx},$$

Hence, in the case where $p = q$, $\hat{\lambda}$ is the first eigenvalue of the problem (P'_λ), *i.e.* the problem $(P'_{\hat{\lambda}})$ has positive solutions for $\lambda \in]0, \hat{\lambda}]$ and has no positive solution for $\lambda > \hat{\lambda}$.

References

1. R. A. Adams, *Sobolev Spaces,* Academic Press, New York, (1975).
2. C. O. Alves, *Existence of Positive Solutions for a Problem with Lack of Compactness Involving the p-Laplacian,* Nonlinear Analysis- TMA, 51 (2002), 1187-1206.
3. C. O. Alves, P. C. Carrião and O. H. Miyagaki, *Signed solution for a class of quasilinear elliptic problem with critical growth,* Commun. Pure Appl. Anal. 1 (2002), no. 4, 531-545.

4. A. Ambrosetti, H. Brezis and G. Cerami, *Combined effects of concave and convex nonlinearities in some elliptic problems,* J. Funct. Anal. 122 (1994), no. 2, 519-543.
5. A. Ambrosetti, J. Garcia Azorero and I.Peral, *Multiplicity results for some nonlinear elliptic equations,* J. Funct. Anal. 137 (1996) no. 1, 219-242.
6. A. Ambrosetti, J. Garcia Azorero and I.Peral, *Existence and multiplicity results for some nonlinear elliptic equations: a survey,* Rend. Mat. Appl. (7) 20 (2000), 167-198.
7. T. Bartsch and M. Willem, *On an elliptic equation with concave and convex nonlinearities,* Proc. Amer. Math. Soc. 123 (1995), no. 11, 3555-3561.
8. J.Berkovits and V.Mustonen, *Nonlinear mappings of monotone type (classification and degree theory),* 1988, Math. Univer. Oulu, Linnanmaa, Oulu, Finland.
9. L. Boccardo, M. Escobedo and I. Peral, *A Dirichlet problem involving critical exponents,* Nonlinear Anal. 24 (1995), 1639-1648.
10. J. F. Bonder and J. D. Rossi, *Existence results for the p-Laplacian with nonlinear boundary conditions,* J. Math. Anal. Appl. 263 (2001) 195-223.
11. J. F. Bonder and J. D. Rossi, *Asymptotic behavior of the best Sobolev trace constant in expanding and contracting domains,* Commun. Pure Appl. Anal. 1 (2002), no. 3, 359-378.
12. P. Dràbek and M. Ôtani, *Global Bifurcation Result for the p-Biharmonic Operator,* Electronic Journal Differential Equations 2001(2001), No 48, pp. 1-19.
13. P. Dràbek and S. Pohozaev, *Positive Solutions for the p-Laplacian: Application of the fibering method,* Proc. Roy. Soc. Edinburgh Sect. A 127 (1997) 703-726.
14. I. Ekeland, *On the Variational Principle,* J. Math. Anal. Appl. 47 (1974) 324-353.
15. A. El Hamidi, *Multiple Solutions with Changing Sign Energy to a Nonlinear Elliptic Equations,* Comm. Pure Appl. Anal. 3 (2004) 253-265.
16. D. Gilbar and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order,* Second ed., Springer New York Tokyo (1983).
17. P. Korman, *On uniqueness of positive solutions for a class of semilinear equations,* Discrete Contin. Dyn. Syst. 8 (2002), no. 4, 865-871.
18. M. Talbi and N. Tsouli, *Existence of Solutions for a Nonlinear Elliptic Problem of Fourth Order with Weight,* Mediterr. j. math. 3 (2006), 87-96.
19. M. Willem, *Minimax Theorems,* Progress in Nonlinear Differential Equations and their Applications, 24. Birkhuser Boston, Inc., Boston, MA, (1996).

A survey on potential theory on Orlicz spaces

N. Aïssaoui

Département de Mathématiques
Ecole Normale Supérieure
B. P. 5206, Ben Souda, Fès Morocco

In this survey we expose the fundamentals of the strongly nonlinear potential theory and relate some of its components. As applications, we establish a relation between this theory and Partial Differential Equations, and show whether the equation $\Delta_A u + h = 0$ possesses a solution or not, for a fixed function h. Here Δ_A is the A-Laplacian which is the p-Laplacian Δ_p, when the Orlicz space $\mathbf{L}_A$ reduces to the Lebesgue space $\mathbf{L}^p$.

Keywords: Orlicz spaces; Capacity; Potentials.

1. Introduction

The nonlinear potential theory on Lebesgue spaces studied by different schools has introduced the notion of capacity in these spaces and has permitted very rich applications in functional analysis, in harmonic analysis, in the theory of probabilities, in the partial differential equations theory, and so forth.

This theory has taken a considerable time for a development in different directions and in different schools. It is one of the most complete theories. Its results permitted, among others, to deepen our knowledge of Sobolev spaces and to solve a great number of problems in higher mathematics. Currently, it makes intervene more general spaces that Sobolev spaces, like Besov spaces and Lizorkin-Triebel spaces. The birth of this theory goes up again to years seventy, to the works of V. G. Maz'ya [75], [76], J. Serrin [89], [90], Yu. G. Reshetnyak [86], and B. Fugled [49], [50], [51], [52], [53]. It took a new thrust during 1970's with notably the works of B. Fugled [54], [55], [56], [57], N. G. Meyers [80], [81], [82], [83], [84], L. I. Hedberg [63], [64], [65], [66], [67], [68], V. G. Maz'ya [77], [78], Yu. G. Reshetnyak [87], T. Sjödin [91], D. R. Adams [1], [2], [3], [4], [5], [6], [7], D. R. Adams and N. G. Meyers [18], [19], D. R. Adams and J. C. Polking [21], V. P. Havin [60],

[61], V. P. Havin and V. G. Maz'ya [62], C. Fernström [48], K. Hansson [59] and others. Other components and applications of this theory are treated in different papers. See for example D. R. Adams [8], [9], [10], [11], [12], [13], D. R. Adams and A. Heard [14], D. R. Adams and L. I. Hedberg [15], D. R. Adams and J. L. Lewis [17], D. R. Adams and M. Pierre [20], H. Aikawa [24], [25], [26], [27], B. Fugled [54], L. I. Hedberg [69], L. I. Hedberg and Th. H. Wolff [70], B. Jawerth, C. Pérez, G. Welland [71], B. O. Turesson [95] and others. The interested reader can consult the indispensable books by D. R. Adams and L. I. Hedberg [16], by W. P. Ziemer [96], by E. M. Stein [92] and by V. G. Maz'ya [79]. We are sure that many important references are missing, because the number of publications is very important, so it is impossible to mention them.

The increasing necessity to work on other spaces to solve other types of equations, says strongly non linears, motivated the creation of a strongly non linear potential theory, that we have introduced in different works. An embryo of this theory is in the thesis of A. Benkirane [46] and in the paper [47] by A. Benkirane and J. P. Gossez. This new theory uses Orlicz spaces and Sobolev-Orlicz spaces that are natural generalizations of Lebesgue and Sobolev spaces. It is important to notice that the main results of the non linear theory spread to the case of strongly non linear one. This new theory doesn't stop marking progress. In this survey we mention some components of this theory, namely

1) the introduction and the study of a capacity in Orlicz spaces, the capacitability of analytic sets and a potential in Orlicz spaces,

2) the study of the continuity of the potential and in particular, the Bessel potential in Orlicz spaces,

3) the study of the instability phenomena of the capacity in Orlicz spaces,

4) the study of some relations between the capacity and maximal operators in Orlicz spaces,

5) the introduction and the study of Wolff inequality in strongly non-linear potential theory and applications,

6) the introduction and the study of the notion of quasicontinuity in Orlicz spaces.

The first relation between the Strongly Nonlinear Potential Theory and Partial Differential Equations is the fact that a compact set K is removable for an elliptic linear operator of order m, with constant coefficients, if and only if its Bessel capacity is null (i.e. $B_{m,A}(K) = 0$).

On the other hand we know that an important application of the non linear potential theory is the resolution of some equations involving the p-Laplacian operator. Hence the p-Laplace equation $\Delta_p u + h = 0$ on $I\!R^N$, $N \leq p$, has no solution if h has a non zero average and the equation $\Delta_p u + g = 0$, on a p-hyperbolic manifold M, has a solution with p-integrable gradient for any bounded measurable function $g : M \to I\!R$ with compact support. In [37] we establish that if the N-function A satisfies the Δ_2 condition and $I\!R^N$ is A-parabolic, then the equation $\Delta_A u + h = 0$ has no weak solution for any function h having a non zero average. Here Δ_A is the A-Laplacian which is the p-Laplacian Δ_p, when the Orlicz space $\mathbf{L}_A$ is the Lebesgue space $\mathbf{L}^p$. We establish also, for a large class of Orlicz spaces $\mathbf{L}_A$ including Lebesgue spaces $\mathbf{L}^p$ $(p > 1)$, that if the function h is in $\mathbf{L}^\infty$ and has a compact support, then the equation $\Delta_A u + h = 0$ has a weak solution when $I\!R^N$ is A-hyperbolic. These results generalize those of V. Gol'dshtein, M. Troyanov in [55] and of M. Troyanov in [91].

We can't develop here other components like the boundedness principle, the weighted strongly nonlinear potential theory or the strongly nonlinear potential theory on metric spaces, because they are out of the scope of this survey. The interested reader can consult [33], [36],]38], [40], [41].

2. Preliminaries

2.1. *Orlicz spaces*

An *N-function* is a continuous convex and even function A defined on $I\!R$, verifying $A(t) > 0$ for $t > 0$,

$$\lim_{t \to 0} \frac{A(t)}{t} = 0 \text{and} \lim_{t \to +\infty} \frac{A(t)}{t} = +\infty.$$

We have the representation $A(t) = \int_0^{|t|} a(x)dx$, where $a : I\!R^+ \to I\!R^+$ is non-decreasing, right continuous, with $a(0) = 0$, $a(t) > 0$ for $t > 0$ and $\lim_{t \to +\infty} a(t) = +\infty$.

The N-function A^* *conjugate* to A is defined by $A^*(t) = \int_0^{|t|} a^*(x)dx$, where a^* is given by $a^*(s) = \sup\{t : a(t) \leq s\}$.

Let A be an N-function and Ω an open set in $I\!R^N$. We note $\mathcal{L}_A(\Omega)$ the set, called an *Orlicz class*, of measurable functions f, on Ω, such that

$$\rho(f, A, \Omega) = \int_\Omega A(f(x))dx < \infty.$$

Let A and A^* be two conjugate N-functions and let f be a measurable function defined almost everywhere in Ω. The *Orlicz norm* of f, $||f||_{A,\Omega}$ or $||f||_A$ if there is no confusion, is defined by

$$||f||_A = \sup\{\int_\Omega |f(x)g(x)|dx : g \in \mathcal{L}_{A^*}(\Omega), \text{ and } \rho(g, A^*, \Omega) \leq 1\}.$$

The set $\mathbf{L}_A(\Omega)$ of measurable functions f such that $||f||_A < \infty$, is called an *Orlicz space*. When $\Omega = I\!R^N$, we set $\mathbf{L}_A$ in place of $\mathbf{L}_A(I\!R^N)$.

The *Luxemburg norm* $|||f|||_{A,\Omega}$ or $|||f|||_A$ if there is no confusion, is defined in $\mathbf{L}_A(\Omega)$ by

$$|||f|||_A = \inf\left\{r > 0 : \int_\Omega A\left(\frac{f(x)}{r}\right) dx \leq 1\right\}.$$

Orlicz and Luxemburg norms are equivalent. More precisely, if $f \in \mathbf{L}_A(\Omega)$, then

$$|||f|||_A \leq ||f||_A \leq 2|||f|||_A.$$

Let A be an N-function. We say that A *verifies the* Δ_2 *condition* if there is a constant $C > 0$ such that $A(2t) \leq CA(t)$ for all $t \geq 0$.

Recall that A verifies the Δ_2 condition if and only if $\mathcal{L}_A = \mathbf{L}_A$. Moreover $\mathbf{L}_A$ is reflexive if and only if A and A^* verify the Δ_2 condition.

We recall the following results. Let A be an N-function and a its derivative. Then

1) A verifies the Δ_2 condition if and only if one of the following holds
   i) $\forall r > 1, \exists k = k(r) : (\forall t \geq 0, A(rt) \leq kA(t))$;
   ii) $\exists \alpha > 1 : (\forall t \geq 0,\ ta(t) \leq \alpha A(t))$;
   iii)$\exists \beta > 1 : (\forall t \geq 0,\ ta^*(t) \geq \beta A^*(t))$;
   iv) $\exists d > 0 : \left(\forall t \geq 0, \left(\frac{A^*(t)}{t}\right)' \geq d\frac{a^*(t)}{t}\right)$.
   Moreover α in *ii*) and β in *iii*) can be chosen such that $\alpha^{-1} + \beta^{-1} = 1$.
   We note $\alpha(A)$ the smallest α such that *ii*) holds.
2) If A verifies the Δ_2 condition, then
   i) $\forall t \geq 1,\ A(t) \leq A(1)t^\alpha$ and $\forall t \leq 1,\ A(t) \geq A(1)t^\alpha$
   ii) $\forall t \geq 1,\ A^*(t) \geq A^*(1)t^\beta$ and $\forall t \leq 1,\ A^*(t) \leq A^*(1)t^\beta$.
   See for instance [47], [73] and [88].

Let A be an N-function such that A and A^* satisfy the Δ_2 condition. We note $\alpha(A) = \alpha$ and $\alpha(A^*) = \alpha^*$. Then we have from 2) above

$$\forall t \geq 0,\ \alpha^* A^*(t) \geq ta^*(t).$$

Hence $\beta \leq \alpha^*$.

If $\beta = \alpha^*$, then $\forall t \geq 0,\ \alpha^* A^*(t) = ta^*(t)$.

This implies that there exists a constant C, such that: $\forall t \geq 0$, $A^*(t) = Ct^{\alpha^*}$.

This means that we are in the case of Lebesgue classes $\mathbf{L}^p$, which is treated in the literature. Hence we suppose in the sequel that $\beta < \alpha^*$.

Let A be an N-function and let $\widehat{A}$ be an N-function equal to A in a neighborhood of infinity and such that (see [23, lemma 4.4]): $\int_0^1 \frac{\widehat{A}^{-1}(t)}{t^{1+\frac{1}{N}}} dt < \infty$. If $\int\limits_1^\infty \frac{\widehat{A}^{-1}(t)}{t^{1+\frac{1}{N}}} dt = \infty$, we define a new N-function $\widehat{A}_1$ by the formula

$$\widehat{A}_1^{-1}(x) = \int\limits_0^x \frac{\widehat{A}^{-1}(t)}{t^{1+\frac{1}{N}}} dt$$

and we let A_1 to be an N-function equal to A in a neighborhood of 0 and to $\widehat{A}_1$ in a neighborhood of infinity (see [23, lemma 4.5] for the construction of such N-function). If $\int\limits_1^\infty \frac{A_1^{-1}(t)}{t^{1+\frac{1}{N}}} dt = \infty$, we start again the same construction and we put $A_2 = (A_1)_1, \dots$.

Let $j = J(A, N)$ be the smallest integer such that $\int\limits_1^\infty \frac{A_j^{-1}(t)}{t^{1+\frac{1}{N}}} dt < \infty$.

If $\int\limits_0^\infty \frac{A^{-1}(t)}{t^{1+\frac{1}{N}}} dt < \infty$, we put $J(A, N) = 0$.

Observe that $J(A, N) \leq N$ because there exists a constant C, such that

$$A^{-1}(t) \leq Ct, \ \forall t \geq 1.$$

Let m be a positive integer. The Orlicz-Sobolev space $W^m\mathbf{L}_A(\Omega)$ is the space of real functions f, such that f and its distributional derivatives up to order m, are in $\mathbf{L}_A(\Omega)$. The space $W^m\mathbf{L}_A(\Omega)$ is a Banach space equipped with the norm:

$$|||f|||_{m,A} = \sum_{|i| \leq m} |||D^i f|||_A, \ f \in W^m\mathbf{L}_A(\Omega).$$

Let $W^{-m}\mathbf{L}_{A^*}(\Omega)$ denote the space of distributions on Ω, which can be written as sums of derivatives up to order m of functions in $\mathbf{L}_{A^*}(\Omega)$. It is a Banach space under the usual quotient norm.

Recall that if A and A^* satisfy the Δ_2 condition, the dual of $W^m\mathbf{L}_A\left(\mathbb{R}^N\right)$ coincides with $W^{-m}\mathbf{L}_{A^*}\left(\mathbb{R}^N\right)$.

For more details on the theory of Orlicz spaces, see [22[, [72], [73], [74] and [88].

2.2. *Capacity and Bessel potentials*

We define a *capacity* as an increasing positive set function C given on a σ-additive class of sets Γ, which contains compact sets and such that $C(\emptyset) = 0$ and $C(\bigcup_{i\geq 1} X_i) \leq \sum_{i\geq 1} C(X_i)$ for $X_i \in \Gamma$, $i = 1, 2, \ldots$.

C is called outer capacity if for every $X \in \Gamma$,

$$C(X) = \inf \{C(O) : O \text{ open, } X \subset O\} .$$

Let k be a positive and measurable function on $I\!R^N$ and let A be an N-function. For $X \subset R^N$, we define

$$C'_{k,A}(X) = \inf\{|||f|||_A : f \in \mathbf{L}_A^+ \text{ and } k * f \geq 1 \text{ on } X\},$$

and

$$C_{k,A}(X) = A\left(C'_{k,A}(X)\right),$$

where $k*\, f$ is the usual convolution. The sign $+$ deals with positive elements in the considered space.

If a statement holds except on a set X where $C_{k,A}(X) = 0$, then we say that the statement holds $C_{k,A}$-*quasieverywhere* (abbreviated $C_{k,A} - q.e$ or $(k, A) - q.e$ if there is no confusion).

We call a function f in $\mathbf{L}_A^+$ such that $k * f \geq 1$ on X, a test function for $C'_{k,A}(X)$. Moreover, a test function, say f, for $C'_{k,A}(X)$ such that $C'_{k,A}(X) = |||f|||_A$ is called a $C'_{k,A}-$capacitary distribution for X and $k * f$ is called a $C'_{k,A}-$capacitary potential for X.

$\boldsymbol{M}$ denotes the vector space of Radon measures. $\boldsymbol{M}_1$ is the Banach space of measures equipped with the norm $||\mu||$ = total variation of $\mu < \infty$.

The space of measures supported by a compact K is denoted by $\boldsymbol{M}(K)$, and the cone of positive elements is $\boldsymbol{M}^+(K)$.

$\mathcal{F}$ will stand for the $\sigma-$field of sets which are $\mu-$measurable for all $\mu \in \boldsymbol{M}_1^+$.

If $\mu \in \boldsymbol{M}_1^+$, we say that μ is concentrated on X if $\mu(y) = 0$ for all sets Y which are $\mu-$measurable and such that $Y \subset {}^cX$.

Let A and A^* be two conjugate N-functions. For $X \in \mathcal{F}$, we define

$$D_{k,A}(X) = \sup\left\{||\mu|| : \mu \in \boldsymbol{M}_1^+,\ \mu \text{ concentrated on } X \text{ and } \ ||k * \mu||_{A^*} \leq 1\right\}$$

where $k * \mu$ is the convolution of k and μ defined by $(k * \mu)(x) = \int k(x - y)d\mu(y)$.

A measure $\mu \in \boldsymbol{M}_1^+$ such that μ is concentrated on X and $||k*\mu||_{A^*} \leq 1$ is called a test measure for $D_{k,A}(X)$. If in addition $D_{k,A}(X) = ||\mu||$, we say that μ is a $D_{k,A}-$capacitary distribution for X and $k*\mu$ is called a $D_{k,A}-$capacitary potential for X.

For $m>0$, the *Bessel kernel* G_m is defined through its Fourier transform $I\!F(G_m)$ as $[I\!F(G_m)](x) = (2\pi)^{-\frac{N}{2}}\left(1+|x|^2\right)^{-\frac{m}{2}}$ where $[I\!F(f)](x) = (2\pi)^{-\frac{N}{2}}\int f(y)e^{-ixy}dy$ for $f \in \mathbf{L}^1$. G_m is positive, in $\mathbf{L}^1$ and verifies the equality: $G_{r+s} = G_r * G_s$.

For more details on Bessel kernels, see [44], [45] and [92].

We note $I_m(x) = |x|^{m-N}$ the Riesz kernel.

We put $B_{m,A} = C_{G_m,A}$, $B'_{m,A} = C'_{G_m,A}$, $R_{m,A} = C_{I_m,A}$ and $R'_{m,A} = C'_{I_m,A}$.

3. Capacity and non linear potential in Orlicz spaces

The essential results concerning the notion of capacity in Orlicz spaces are resumed in the following theorem; see [28] and [42].

Theorem 3.1. *Let A be an N-function. Then*

a) $C'_{k,A}$ is an outer capacity defined on all subsets of $I\!R^N$.
b) $C'_{k,A}(X) = 0$ if and only if there exists $f \in \mathbf{L}_A^+$ such that $k*f = +\infty$.
c) If $f_n \to f$ strongly in $\mathbf{L}_A$, then

(1) $k*f_n \to k*f$ in $C'_{k,A}-$capacity,

(2) there exists a subsequence $(f"_n)_n$ of the sequence $(f_n)_n$ such that

$$k*f"_n \to k*f \quad C'_{k,A}\text{-quasi uniformly,}$$

(3) $k*f"_n \to k*f$ $C'_{k,A}$-quasi everywhere.

d) If $(K_n)_n$ is a decreasing sequence of compact set and $K = \cap_n K_n$, then

$$\lim_{n\to\infty} C'_{k,A}(K_n) = C'_{k,A}(K_n).$$

e) If $(X_n)_n$ is a increasing sequence of sets in $I\!R^N$ and if $\mathbf{L}_A$ is a reflexive Orlicz space, then $\lim_{n\to\infty} C'_{k,A}(X_n) = C'_{k,A}(\cup_n X_n)$.
f) If $\mathbf{L}_A$ is a reflexive Orlicz space, then analytic sets are $C'_{k,A}$-capacitable.
g) If A is an N-function and $D^*_{k,A}$ is the outer capacity associated to $D_{k,A}$, then for any set X, $D^*_{k,A}(X) = C'_{k,A}(X)$.
h) If $\mathbf{L}_A$ is a reflexive Orlicz space, then for all analytic sets X

$$D_{k,A}(X) = C'_{k,A}(X).$$

Let A be an N-function and k a positive and measurable function defined on $I\!R^N$. Let X be any set such that $C'_{k,A}(X) < \infty$. We consider the following variational problem: Find $f_0 \in \mathbf{L}_A^+$ such that $k * f_0 \geq 1$ $C'_{k,A}$-q.e. on X and

$|||f_0|||_A = \inf\{|||f|||_A : f \in \mathbf{L}_A{}^+,\ k * f \geq 1\ C'_{k,A}\text{-q.e. on } X\}$.

An answer is given in the following theorem; see [43].

Theorem 3.2. *1) Let $\mathbf{L}_A$ be a reflexive Orlicz space and X be any set such that $C'_{k,A}(X) < \infty$. Then X has a unique $C'_{k,A}$-capacitary distribution f; $f \in \mathbf{L}_A^+$, $k * f \geq 1$ on X and $C'_{k,A}(X) = |||f|||_A$.*

2) Let $\mathbf{L}_A$ be a reflexive Orlicz space and X be an analytic set. If γ is the $D_{k,A}-$capacitary distribution for X and f is the $C'_{k,A}$-capacitary distribution for X, then

$$k * \gamma = \left[\left\|a \circ (f\ .|||f|||_A^{-1})\right\|_{A^*}\right]^{-1} a \circ (f\ .|||f|||_A^{-1})\ \ a.e.$$

Moreover $k * f \leq 1$ on suppγ, where a is the derivative of A.

4. On the Bessel potentials in Orlicz spaces

4.1. *On the continuity of Bessel potentials in Orlicz spaces*

The results in this subsection show that Bessel capacities in reflexive Orlicz spaces are non increasing under orthogonal projection of sets; see [29]. This is used to get a continuity of potentials on some subspaces. The obtained results generalize those of Meyers and Reshetnyak in the case of Lebesgue classes.

Theorem 4.1. *1) Let A be an N-function such that A and A^* satisfy the Δ_2 condition. Let k be a kernel on $I\!R^N$ which is spherically symmetric and non-increasing as $|x|$ increases. If S is an affine subspace of $I\!R^N$ and X a subspace of $I\!R^N$, then*

$$C_{k,A}\left(P_S X\right) \leq C_{k,A}\left(X\right).$$

2) Let A be an N-function such that A and A^* satisfy the Δ_2 condition. Let k be a kernel on $I\!R^N$ which is spherically symmetric and non-increasing as $|x|$ increases. Further, suppose that k is locally integrable with $\lim_{|x|\to\infty} k(x) = 0$. Let S be an affine subspace of $I\!R^N$. Then

a) For $f \in \mathbf{L}_A$ and $\varepsilon > 0$, there exists a closed set $F \subset S$ such that

$$C_{k,A}(S - F) < \varepsilon \text{ and } k * f \in C_0(F + S^\perp).$$

Hence

$$k * f \in C_0(x + S^\perp)\ \ C_{k,A} - q.e. \text{ in } S.$$

b) Let $(f_i)_i$ be a sequence convergent to f in $\mathbf{L}_A$. Then there is a subsequence $(f_{i'})_{i'}$, such that given $\varepsilon > 0$, there exists a closed set $F \subset S$ with the property

$$C_{k,A}(S - F) < \varepsilon \text{ and } k * f_{i'} \to k * f \text{ in } C_0(F + S^{\perp}).$$

Hence

$$k * f_{i'} \to k * f \text{ in } C_0(x + S^{\perp}) \quad C_{k,A} - q.e. \text{ in } S.$$

Theorem 4.2. *Let A be an N-function such that A and A^* satisfy the Δ_2 condition. Let S be an affine subspace of $\mathbb{R}^N$ and $0 \le s < m$. Then*

(1) For $f \in \mathbf{L}_A$ and $\varepsilon > 0$, there exists a closed set $F \subset S$ such that

$$B_{m-s,A}(S - F) < \varepsilon$$

and

$$D^j(G_m * f) \in C_0(F + S^{\perp}) \text{ for all } j,\ |j| \le s.$$

Hence for such j,

$$D^j(G_m * f) \in C_0(x + S^{\perp}) \quad B_{m-s,A} - q.e. \text{ in } S.$$

(2) Let $(f_i)_i$ be a sequence convergent to f in $\mathbf{L}_A$. Then there is a subsequence $(f_{i'})_{i'}$, such that given $\varepsilon > 0$, there exists a closed set $F \subset S$ with the property

$$B_{m-s,A}(S - F) < \varepsilon$$

and

$$D^j(G_m * f_{i'}) \to D^j(G_m * f\ ) \text{ in } C_0(F + S^{\perp}) \text{ for all } j,\ |j| \le s.$$

Hence for such j,

$$D^j(G_m * f_{i'}) \to D^j(G_m * f\ ) \text{ in } C_0(x + S^{\perp}) \quad B_{m-s,A} - q.e. \text{ in } S.$$

Theorem 4.3. *Let A be an N-function and T be a one to one map of $\mathbb{R}^N$ onto itself. Suppose that T and its inverse T^{-1} satisfy a Lipschitz condition. Let ρ, $0 < \rho < \infty$, and $X \subset \mathbb{R}^N$ be such that diam $X \le \rho$. Then there exists a constant C, independent of X such that*

$$B'_{m,A}\left[T(X)\right] \le C B'_{m,A}(X).$$

If $I\!R^N$ is the affine direct sum of G and H, we define P_{GH} as the projection of $I\!R^N$ onto G, parallel to H.

Theorem 4.4. *Let A be an N-function such that A and A^* satisfy the Δ_2 condition. Let ρ, $0 < \rho < \infty$ and $X \subset I\!R^N$ be such that diam $P_{GH}X \leq \rho$. Then there exists a constant C, independent of X such that*

$$B'_{m,A}(P_{GH}X) \leq CB'_{m,A}(X).$$

4.2. *Instability of capacity in Orlicz spaces*

In this subsection we describe the instability of certain capacities in Orlicz spaces. Our results generalize some of those obtained by C. Fernström in [48] and the Theorem 9 of Hedberg in [7] in the case of Lebesgue spaces. See [30].

For $m > 0$ we define the Riesz kernel, I_m, by $I_m(x) = |x|^{m-N}$ for $x \in I\!R^N$.

We suppose that $m < N$. If A is an N-function and X any subset of $I\!R^N$, we define $R'_{m,A}(X)$ as

$$R'_{m,A}(X) = \inf\{|||f|||_A : f \in \mathbf{L}^+_A \text{ and } I_m * f \geq 1 \text{ on } X\}.$$

We know that $R'_{m,A}$ is an outer capacity; see [40]. We let $R_{m,A} = A \circ R'_{m,A}$.

Definition 4.1. Let X be a Borel set and $\mathbf{m}$ be the Lebesgue measure on $I\!R^N$. Then x is a density point for X if

$$\lim_{r\to 0} \mathbf{m}(B(x,r))^{-1}\, \mathbf{m}(X \cap B(x,r)) = 1.$$

Theorem 4.5. *Let A be an N-function verifying the Δ_2 condition. Suppose that $m < \frac{N}{\alpha}$. Let X be a Borel set and x a density point for X. Then*

$$\lim_{r\to 0} \left[R'_{m,A}(B(x,r))\right]^{-1} .R'_{m,A}(X \cap B(x,r)) = 1.$$

Corollary 4.1. *Let A be an N-function verifying the Δ_2 condition. Suppose that $m < \frac{N}{\alpha}$. Let X be a Borel set. Then*

$$\lim_{r\to 0} \left[R'_{m,A}(B(x,r))\right]^{-1} .R'_{m,A}(X \cap B(x,r)) = 1, \text{ a.e. on } X.$$

Theorem 4.6. *Let A be an N-function verifying the Δ_2 condition and E be a Borel set. Suppose that $m < \frac{N}{\alpha}$. Then a.e. on $I\!R^N$ one of the following relations holds*

$$\lim_{r\to 0} \left[R'_{m,A}(B(x,r))\right]^{-1} .R'_{m,A}(E \cap B(x,r)) = 1$$

or

$$\lim_{r\to 0} \lim r^{-N}.R_{m,A}(E\cap B(x,r)) = 0.$$

Theorem 4.7. *Let A be an N-function verifying the Δ_2 condition and E be an everywhere dense Borel set. Suppose that $m < \frac{N}{\alpha}$. Then the following are equivalent.*

(i) $R_{m,A}(E\cap O) = R_{m,A}(O)$ *for every open O.*
(ii) $R_{m,A}(E\cap B(x,r)) = R_{m,A}\left(B(x,r)\right)$ *for all x and r.*
(iii) *For almost all x (with respect to Lebesgue measure)*

$$\lim_{r\to 0} \sup r^{\frac{-N}{\alpha}} R'_{m,A}(E\cap B(x,r)) > 0.$$

The next Theorem gives a connection between Riesz and Bessel capacities in the case of Orlicz spaces.

Theorem 4.8. *(a) Let m be a positive number such that $m < N$. Then there exists a constant δ, such that for all N-functions A and all sets $X \subset \mathbb{R}^N$,*

$$R'_{m,A}\left(X\right) \leq \delta B'_{m,A}\left(X\right).$$

(b) Let A be an N-function verifying the Δ_2 condition such that $m < \frac{N}{\alpha}$. Let B_r be the ball centered at 0 with radius r. Then for all sets $X \subset B_r$, there exists a constant δ, independent of X, but dependent on r, such that

$$B'_{m,A}\left(X\right) \leq \delta R'_{m,A}\left(X\right).$$

Remark 4.1. Since A verifies the Δ_2 condition, there exists a constant γ', independent of X, but dependent of r, such that

$$B_{m,A}\left(X\right) \leq \gamma' R_{m,A}\left(X\right).$$

4.3. *Bessel potentials in Orlicz spaces*

In this subsection it is shown that Bessel potentials have a representation in term of measure when the underlying space is Orlicz. A comparison between capacities and Lebesgue measure is given and geometric properties of Bessel capacities in this space are studied. Moreover it is shown that if the capacity of a set is null, then the variation of all signed measures of this set is null when these measures are in the dual of an Orlicz-Sobolev space; see [31].

4.3.1. *Representation of potentials and comparison with Lebesgue measure*

Theorem 4.9. *Let A be an N-function such that A and A^* satisfy the Δ_2 condition. Let m be a positive integer and X a set in $I\!R^N$ such that $0 < B'_{m,A}(X) < \infty$.*

Let f be the $B'_{m,A}-$capacitary distribution of X. Then there exists a positive measure μ_X such that:

1) $G_m * f = B'_{m,A}(X).G_m * \left[a^{-1} \circ (G_m * \mu_X)\right]$, *where a is the derivative of A.*
2) *supp* $\mu_X \subset \overline{X}$.

   If in addition we suppose that X is compact, then
3) $\mathcal{G}_m * f \leq 1$ *on supp* μ_X.

Theorem 4.10. *Let A be an N-function such that A and A^* satisfy the Δ_2 condition. Let m be a positive integer. Then*

(1) *If $m \leq J(A, N)$ there exists a constant $C = C(A, N, m) > 0$ such that*

$$B'_{m,A}(X) \geq C\left[A_m^{-1}\left(\frac{1}{\mathbf{m}^*(X)}\right)\right]^{-1}$$

for all set X such that $\mathbf{m}^*(X) \neq 0$. (Here $\mathbf{m}$ is the Lebesgue measure on $I\!R^N$ and $\mathbf{m}^*$ is the outer measure associated to $\mathbf{m}$).

(2) *If $m > J(A, N)$, there exists a constant $C = C(A, N, m) > 0$ such that for all set $X \neq \emptyset$,*

$$B'_{m,A}(X) \geq C.$$

Theorem 4.11.

(1) *Let A be an N-function and m be such that $0 < m < N$. Let $S_\rho = B(x, \rho)$ be the open ball centered at x and with radius ρ. Then there exists a constant C independent of ρ such that*

$$B'_{m,A}(S_\rho) \leq C\rho^{-m} \text{ for } 0 < \rho \leq 1.$$

(2) *Let A be an N-function satisfying the Δ_2 condition and let m be such that $0 < m < N$. Let $C(A)$ be the smallest constant C' such that: $A(2t) \leq C'A(t)$, $\forall t$.*

Then there exists a constant C independent of ρ such that: $B'_{m,A}(S_\rho) \leq C2^{-q}\rho^{-m}$ for $0 < \rho \leq 1$, where q is the greatest positive integer such that $q \leq \frac{Log\rho^{-N}}{LogC(A)}$.

4.3.2. *Relation between capacity and Hausdorff measure*

Theorem 4.12. *Let A be an N-function such that A and A^* satisfy the Δ_2 condition. Let m be a positive real such that $\alpha m < N$, where $\alpha = \alpha(A)$. Let $\widehat{u}$ be a positive, decreasing function defined on $I\!R$, continuous from the right and such that*

$$G_m(r) \leq \widehat{u}(r) \text{ and } \lim_{r \to 0} \widehat{u}(r) G_m(r)^{-1} = +\infty.$$

If $B' = B'_{\widehat{u},A}$, then $\lim_{\rho \to 0} B'(S_\rho) B'_{m,A}(S_\rho)^{-1} = 0$.
In particular, if $\alpha m < N$, then $\lim_{\rho \to 0} B'_{m,A}(S_\rho) = 0$.

Definition 4.2. Let $\varphi(r)$ be a positive, increasing function in some interval $[0, r'[$ and such that $\lim_{r \to 0} \varphi(r) = 0$. If X is an arbitrary set, the Hausdorff φ-measure of X is given by

$$H_{\varphi(r)}(X) = \lim_{s \to 0} (\inf \sum_{i \geq 1} \varphi(r_i)),$$

where the above infimum is taken over all countable coverings of X by spheres $S(x_i, r_i)$ such that $r_i \leq s$.

Note that $H_{\varphi(r)}$ is a capacity which has the property

$$H_{\varphi(r)}(X) = H_{\varphi(r)}(Y),$$

where Y is a G-set containing X.

Theorem 4.13. *Let A be an N-function such that A and A^* satisfy the Δ_2 condition and let X be a subset of $I\!R^N$. Let m be a positive real such that $\alpha m < N$, where $\alpha = \alpha(A)$ and let $\varphi(r) = B'_{m,A}(S_r)$. Then*

$$B_{m,A}(X) = 0 \text{ if } H_{\varphi(r)}(X) < \infty.$$

4.3.3. *Capacities and measures in Orlicz-Sobolev spaces*

Theorem 4.14. *1) Let A be an N-function such that A and A^* satisfy the Δ_2 condition and let m be a positive integer such that $m \leq J(A, N)$. Let $T \in W^{-m}\mathbf{L}_{A^*}\left(I\!R^N\right) \cap \mathbf{M}_1\left(I\!R^N\right)$ and let K be a compact set such that $B_{m,A}(K) = 0$ and $T^-(K) = 0$.*

Then $\|T\|(K) = 0$.

2) Let A be an N-function such that A and A^* satisfy the Δ_2 condition and let m be a positive integer such that $m \leq J(A, N)$. Let $T \in W^{-m}\mathbf{L}_{A^*}\left(I\!R^N\right) \cap \mathbf{M}_1\left(I\!R^N\right)$ and let X be a $\|T\|$-measurable set such that $B_{m,A}(X) = 0$ and $T^-(X) = 0$.

Then $\|T\|(X) = 0$.

3) Let A *be an* N*-function such that* A *and* A^* *satisfy the* Δ_2 *condition and let* m *be a positive integer such that* $m \leq J(A,N)$*. Let* $T \in W^{-m}\mathbf{L}_{A^*}\left(\mathbb{R}^N\right) \cap \mathbf{M}_1\left(\mathbb{R}^N\right)$ *and* K *be a compact set such that* $B_{m,A}(K) = 0$*. Then* $\|T\|(K) = 0$.

4) Let A *be an* N*-function such that* A *and* A^* *satisfy the* Δ_2 *condition and let* m *be a positive integer such that* $m \leq J(A,N)$*. Let* $T \in W^{-m}\mathbf{L}_{A^*}\left(\mathbb{R}^N\right) \cap \mathbf{M}_1\left(\mathbb{R}^N\right)$ *and let* X *be a set such that* $B_{m,A}(X) = 0$*. Then* $\|T\|(X) = 0$.

5. Another developments of strongly nonlinear potential theory

We establish some relations between potentials and maximal functions in Orlicz spaces. We give a definition of quasicontinuity and obtain a description of quasicontinuous representative in some potential spaces. We also give a result on smooth truncation of potentials in Orlicz-Sobolev spaces and compare some capacities. As a consequence: a compact is removable in Orlicz space for an elliptic linear operator of order m with constant coefficients if and only if its Bessel capacity of order m is null. See [32].

5.1. *Maximal operators and potentials*

Let f be a locally integrable function. The Hardy-Littlewood maximal function associated to f is defined by $Mf(x) = M_0 f(x) = \sup_{r>0} |B(x,r)|^{-1} \int\limits_{B(x,r)} |f(y)|\, dy$.

Here $|B(x,r)|$ is the Lebesgue measure of $B(x,r)$ on $\mathbb{R}^N$.

The fractional maximal function associated to f is defined for $0 < \alpha < N$, by $M_\alpha f(x) = \sup_{r>0} |B(x,r)|^{\frac{\alpha-N}{N}} \int_{B(x,r)} |f(y)| dy$.

And for $0 \leq \alpha < N$, and $\delta > 0$, the inhomogeneous version of these functions is defined by $M_{\alpha,\delta} f(x) = \sup_{\delta \geq r > 0} |B(x,r)|^{\frac{\alpha-N}{N}} \int\limits_{B(x,r)} |f(y)|\, dy$.

For $0 \leq \alpha < N$, and $\delta > 0$, we define the modified Riesz kernel, $I_{\alpha,\delta}$ by

$$I_{\alpha,\delta}(x) = I_\alpha(x),\ if\ \ |x| < \delta,$$
$$I_{\alpha,\delta}(x) = 0,\ if\ \ |x| \geq \delta.$$

The Riesz potential $I_\alpha * \mu$, $0 < \alpha < N$, where μ is a positive measure, can be estimated below by the fractional maximal function associated to μ. In fact, for every $r > 0$,

$$\int_{R^N} |x-y|^{\alpha} - N d\mu(y) \geq \int_{|x-y|\leq r} |x-y|^{\alpha-N} d\mu(y) \geq \int_{|x-y|\leq r} d\mu(y).$$

The reverse inequality is false in general.

In the first part of the following theorem, we give a generalization to Orlicz spaces, of the classical theorem of B. Muckenhoupt and R.I. Wheeden [85], which establishes the opposite inequality in term of L^p norms.

Theorem 5.1. *Let A be an N-function satisfying the Δ_2 condition, and let $0 < \alpha < N$. Then*

(1) There is a constant $C > 0$, such that for any positive measure μ,
i) $C^{-1} \int A(M_\alpha \mu)dx \leq \int A(I_\alpha * \mu)dx \leq C \int A(M_\alpha \mu)dx$
ii) $C^{-1}|||M_\alpha \mu|||_A \leq |||I_\alpha * \mu|||_A \leq C|||M_\alpha \mu|||_A$.

(2) If δ is a positive number, there are positive constants C_1, C_2 and C_3 such that for any positive measure μ,
$|||M_{\alpha,\delta}\, \mu|||_A \leq C_1|||I_{\alpha,\delta} * \mu|||_A \leq C_2|||G_\alpha * \mu|||_A \leq C_3|||M_{\alpha,\delta}\, \mu|||_A$.

5.2. *Quasicontinuity*

We recall the general definition of quasicontinuity.

Definition 5.1. Let $\mathcal{C}$ be a capacity on $I\!R^N$ and let f be a function defined $\mathcal{C}-$q.e. on $I\!R^N$ or on some open subset of $I\!R^N$. Then f is said to be $\mathcal{C}-$quasicontinuous if for every $\epsilon > 0$, there is an open set O such that $\mathcal{C}(O) < \epsilon$ and $f\mid_{O^c} \in C(O^c)$.

In other words, the restriction of f to the complement of O is continuous in the induced topology.

For Bessel capacity $B'_{m,A}$, we write (m,A)-quasicontinuous in place of $B'_{m,A}$-quasicontinuous.

Theorem 5.2. *1) Let A be an N-function satisfying the Δ_2 condition. If $f \in \mathbf{L}_A$, then the potential $G_m * f$, $m > 0$, is $(m,A)-$quasicontinuous.*

Hence every element in $\mathbf{L}_{m,A}$ has an $(m,A)-$quasicontinuous representative.

2) Let A be any N-function. Let $f = G_m * g \in \mathbf{L}_{m,A}$, $m > 0$. Then $\lim_{r\to 0} \frac{1}{|B(x,r)|} \int_{B(x,r)} f(y)dy = G_m * g(x)$, *wherever* $G_m * |g|(x) < \infty$, *i.e., $(m,A) - q.e.$*

Theorem 5.3. *1) Let A be any N-function. Let f_1 and f_2 be two $(m,A)-$quasicontinuous functions, $m > 0$. Suppose that $f_1(x) = f_2(x)$ almost everywhere. Then*

$f_1(x) = f_2(x)$ $(m, A)-$quasieverywhere.

2) Let A be an N-function satisfying the Δ_2 condition. Let $f, g \in \mathbf{L}_{m,A}$, $m > 1$. Then $D(fg) = f(x)Dg(x) + Df(x)g(x)$ $(m-1, A) - q.e.$

We extend the last theorem for any kernel and with a less restrictive property that the quasicontinuity; but for reflexive Orlicz space. This is the tribute to pay!

By $\mathbf{B}(I\!\!R^N)$ we note the family of Borelian sets in $I\!\!R^N$.

Theorem 5.4. *Let k be any kernel and suppose that the N-function A is such that A and A^* satisfy the Δ_2 condition. Let g_1 and g_2 be two functions verifying the following: $\forall \epsilon > 0$, $\exists X \in \mathbf{B}(I\!\!R^N) : C_{k,A}(X) < \epsilon$ and the restrictions of g_1 and g_2 to ${}^cX$ are continuous.*

Suppose that $\{x : g_1(x) \neq g_2(x)\} \in \mathbf{B}(I\!\!R^N)$ and that $g_1(x) = g_2(x)$ a.e. Then

$$g_1(x) = g_2(x)(k, A) - q.e.$$

5.3. *Operations on potentials, Other definition of capacity and removable singularities*

For reflexive Orlicz spaces, we establish that composition of Bessel potential with a smooth operator, is a potential. This is an extension of a well known Theorem of V.G. Maz'ya which is a substitute of the fact that the Sobolev spaces $W^{m,p}(m \neq 1)$ are not closed under contractions. An immediate consequence is the equivalence of capacities $N_{m,A}$ and $B_{m,A}$. Note that in the case of L^p Lebesgue spaces, this two capacities are equivalent even if m is not integer. See [21]. The correspondent case for Orlicz spaces remains open.

On the other hand, we show that in reflexive Orlicz spaces, a compact set K is removable for an elliptic linear operator of order m, with constant coefficients, if and only if its Bessel capacity is null (i.e. $B_{m,A}(K) = 0$). This is the first relation between the Strongly Nonlinear Potential Theory, and Partial Differential Equations.

Theorem 5.5. *Let m be an integer such that $0 < m < N$ and A be an N-function such that A and A^* satisfy the Δ_2 condition. Let k be an integer such that $k \geq m$ and $T \in \mathbf{C}^k(I\!\!R^+)$ verifies the following condition*

$\sup \left| x^{i-1} T^{(i)}(x) \right| \leq L < \infty$, $i = 1, 2, ..., k$.

Then $T \circ (G_m * f) \in \mathbf{L}_{m,A}$, for all $f \in \mathbf{L}_A^+$, and there is a constant C, which depends only on A, m and N, such that

$|||T \circ (G_m * f)|||_{m,A} \leq CL|||G_m * f|||_{m,A} = CL|||f|||_A$.

Definition 5.2. For $X \subset I\!R^N$, we pose

$$N_{k,A}(X) = \inf\{A(|||\varphi|||_{k,A}) : \varphi \in \mathbf{S} \text{ and } \varphi = 1 \text{ in a neighborhood of } X\}$$

$$N'_{k,A}(X) = \inf\{|||\varphi|||_{k,A} : \varphi \in \mathbf{S} \text{ and } \varphi = 1 \text{ in a neighborhood of } X\}.$$

Here $\mathbf{S} = \mathbf{S}(I\!R^N)$ is the Schwartz space of rapidly decreasing functions.

If $k = G_m$, we write $N_{m,A} = N_{G_m,A}$.

Definition 5.3. Let $K \subset I\!R^N$ be a compact set, and let $\mathbf{P}$ a partial differential operator defined in a neighborhood of K. Then K is said to be removable for $\mathbf{P}$ in $\mathbf{L}_A$ if any solution v of $\mathbf{P}v = 0$ in $O \backslash K$ for some bounded open neighborhood of K, such that $v \in \mathbf{L}_A(O \setminus K)$, can be extended to a function $\widetilde{v} \in \mathbf{L}_A(O)$ such that $\mathbf{P}\widetilde{v} = 0$ in O.

Theorem 5.6. *Let m be an integer such that $0 < m < N$. Let $K \subset I\!R^N$ be a compact set, and let $\mathbf{P}$ an elliptic linear partial differential operator of order m with constant coefficients. Let A be an N-function such that A and A^* satisfy the Δ_2 condition. Then K is not removable for $\mathbf{P}$ in $\mathbf{L}_A$ if $B_{m,A^*}(K) > 0$, and it is removable if $N_{m,A^*}(K) = 0$.*

We remark the immediate inequality $B_{m,A}(X) \leq N_{m,A}(X)$. In view of the last theorem, it is of considerable interest that these set functions are in fact equivalent.

Theorem 5.7. *Let m be an integer such that $0 < m < N$, and A be an N-function such that A and A^* satisfy the Δ_2 condition. Then there is a constant C such that for all $X \subset R^N$, $B_{m,A}(X) \leq N_{m,A}(X) \leq CB_{m,A}(X)$.*

This means that a compact $K \subset I\!R^N$, is removable in $\mathbf{L}_A$ for an elliptic linear operator of order m with constant coefficients if and only if $B_{m,A}(K) = 0$.

The first L^p version of this theorem was been proved by V.G. Maz'ya [79, Chapter 9.3]. The L^p version for general m is due to D.R. Adams and J.C. Poking [21]. The general case when $m \in I\!R$ is such that $0 < m < N$, remains open.

6. Capacitary type estimates in strongly nonlinear potential theory and applications

In this section general result on smooth truncation of Riesz and Bessel potentials in Orlicz-Sobolev spaces is given and a capacitary type estimate is presented. We construct also a space of quasicontinuous functions and an alternative characterization of this space and a description of its dual are established. For the Riesz kernel I_m, we get that operators of strong type $(A,\, A)$, are also of capacitaries strong and weak types $(m,\, A)$. See [34].

6.1. *A capacitary type estimate*

Theorem 6.1. *Let A be an N-function such that A and A^* verify the Δ_2 condition, $\alpha = \alpha(A)$ and m is a positive integer. Let $T_{j\in\mathbf{Z}}$ be a doubly infinite sequence of $C^m(I\!R)$ functions identically zero for $t<0$ with T_j' having disjoints supports in $(0,\infty)$ and such that*

$$\sup_{t>0}\left|t^{k-1}T_j^{(k)}(t)\right| \le L < \infty, k = 0, 1, ..., m.$$

Then for all $f \in \mathbf{L}_A^+$, there is a constant C depending only on N, m, L and A such that

$$\sum_j |||D^\beta T_j(\mathcal{S}_m * f)|||_A \le C|||f|||_A,$$

where β is a multi-index such that $|\beta| = m$, and $\mathcal{S}_m$ is either G_m if m is a positive integer, or I_m if m is a positive integer such that $m < N/\alpha$.

Let $\mathcal{S}_m$ as above and set $\mathcal{S}'_{m,A}(X) = \inf\{|||f|||_A : f \in \mathbf{L}_A^+ \text{ and } \mathcal{S}_m * f \ge 1 \text{ on } X\}$.

Theorem 6.2. *Let A be an N-function such that A and A^* verify the Δ_2 condition and m a positive integer. Then there is a constant C depending only on N, m and A such that for all $f \in \mathbf{L}_A^+$*

$$\textstyle\int_0^\infty \mathcal{S}'_{m,A}(\{x : \mathcal{S}_m * f(x) \ge t\})dt \le C|||f|||_A.$$

6.2. *A space of quasicontinuous functions*

This subsection is devoted to generalize some results in [6] and in [55] relative to the $\mathbf{L}^p$ Lebesgue classes. From the previous Theorem, it is natural to seek when the quantity

$$\int_0^\infty B'_{m,A}\left(\{x : |\psi| \ge t\}\right) dt \tag{1}$$

defines a norm on a linear space of functions ψ on $I\!R^N$. The answer is not known in general, but we establish that (1) is equivalent to a certain norm of ψ.

Definition 6.1. For ψ a function on $I\!R^N$, define for $m > 0$, $\mathbf{K}_\psi$ and $\Lambda_{m,A}(\psi)$ as

$$\mathbf{K}_\psi = \left\{f \in \mathbf{L}_A^+ : G_m * f(x) \geq |\psi(x)|\,, \forall x \in I\!R^N\right\}$$
$$\Lambda_{m,A}(\psi) = \inf\left\{|||f|||_A : f \in \mathbf{K}_\psi\right\}.$$

Theorem 6.3. *1. Let A be any N-function and $m \in I\!R^+$. The function $\Lambda_{m,A}$ defines a norm on $C_0 = C_0(I\!R^N)$, and*

$$\frac{1}{4}\Lambda_{m,A}(\psi) \leq \int_0^\infty B'_{m,A}\left(\{x : |\psi(x)| \geq t\}\right) dt$$

for all continuous functions ψ.

2. Let A be an N-function such that A and A^* verify the Δ_2 condition and m a positive integer. Then there is a constant C such that

$$\frac{1}{4}\Lambda_{m,A}(\psi) \leq \int_0^\infty B'_{m,A}\left(\{x : |\psi(x)| \geq t\}\right) dt \leq C\Lambda_{m,A}(\psi)$$

for all continuous functions ψ.

We define a new Banach space, $\mathbf{L}_A(B'_{m,A})$, as the completion of $D(I\!R^N)$ in the norm $\Lambda_{m,A}$.

Theorem 6.4. *1) Let A be an N-function such that A and A^* verify the Δ_2 condition and m a positive integer. Then $\mathbf{L}_{m,A} \subset \mathbf{L}_A(B'_{m,A})$ and any continuous compactly supported function belongs on $\mathbf{L}_A(B'_{m,A})$.*

2) Let A be an N-function such that A and A^* verify the Δ_2 condition and m a positive integer. If $\psi \in C$ is such that $\int_0^\infty B'_{m,A}\left(\{x : |\psi(x)| \geq t\}\right) dt < \infty$, then $\psi \in \mathbf{L}_A(B'_{m,A})$.

Theorem 6.5. *Let A be an N-function such that A and A^* verify the Δ_2 condition and m a positive integer. Then a function ψ on $I\!R^N$ belongs to $\mathbf{L}_A(B'_{m,A})$ if and only if it is (m, A)-quasicontinuous, and*

$$\int_0^\infty B'_{m,A}\left(\{x : |\psi(x)| \geq t\}\right) dt < \infty.$$

Corollary 6.1. *Let A and m be as in the previous theorem. If $\psi \in \mathbf{L}_A(B'_{m,A})$, and if φ is an (m, A)-quasicontinuous function such that $|\varphi| \leq |\psi|$ a.e., then*

$\varphi \in \mathbf{L}_A(B'_{m,A})$, and $\|\varphi\|_{\mathbf{L}_A(B'_{m,A})} \leq \|\psi\|_{\mathbf{L}_A(B'_{m,A})}$.

Moreover, if $\mathbf{L}_{m,A}$ is imbedded in a Banach space $\mathbf{B}$ such that $\|.\|_\mathbf{B}$ is monotone in the sense that $\|u\|_\mathbf{B} \leq ||v||_\mathbf{B}$ for all u and v such that $|u(x)| \leq |v(x)|$ everywhere, then $\mathbf{B}$ contains $\mathbf{L}_A(B'_{m,A})$.

We describe now the dual space to $\mathbf{L}_A(B'_{m,A})$.

Theorem 6.6. *Let A be an N-function such that A verifies the Δ_2 condition and $m > 0$. Then the dual space $\mathbf{L}_A(B'_{m,A})^*$ can be identified with the space of all $\mu \in \mathbf{M}(I\!R^N)$ such that $G_m * |\mu| \in \mathbf{L}_{A^*}$. If $\psi \in \mathbf{L}_A(B'_{m,A})$ and $\mu \in \mathbf{L}_A(B'_{m,A})^*$, then $\psi \in \mathbf{L}^1(|\mu|)$, and the duality is given by*

$$\langle \mu, \psi \rangle = \int_{R^N} \psi d\mu.$$

Moreover, the norm of μ in $\mathbf{L}_A(B'_{m,A})^*$ is $||G_m * |\mu|\,||_{A^*}$.

6.3. *Maximal operators and capacity*

Let $(\theta_j)_j$ be a sequence of convolution operators. Define the *maximal operator J* by $J(f) = \sup_j |\theta_j * f|$, where f is initially taken to be in the Schwarz class of rapidly decreasing C^∞ functions on $I\!R^N$ denoted by $S = S(I\!R^N)$.

An operator $H : \mathbf{L}_A \to \mathbf{L}_A$ is of strong type $(A,\, A)$ if

$$|||H(f)|||_A \le C|||f|||_A, \forall f \in \mathbf{L}_A,$$

where C is a constant dependent only on A.

For more details, see [93].

Definition 6.2. An operator $H : \mathbf{L}_A \to \mathbf{L}_A$ is of capacitary weak type $(A,$ $A)$ if

$$\forall f \in \mathbf{L}_A, \forall t > 0,\; R'_{m,A}\left(\{x : H(I_m * f)(x) \ge t\}\right) \le C_A \tfrac{|||f|||_A}{t},$$

where C_A is a constant dependent only on N, m and A.

H is of capacitary strong type $(m,\, A)$ if

$$\forall f \in \mathbf{L}_A, , \textstyle\int_0^\infty R'_{m,A}\left(\{x : H(I_m * f)(x) \ge t\}\right) dt \le C|||f|||_A,$$

where C is a constant dependent only on N, m and A.

Theorem 6.7. *1. Let A be an N-function such that A and A^* verify the Δ_2 condition, $\alpha = \alpha\,(A)$ and m is a positive integer such that $m < N/\alpha$. If J is of strong type $(A,\, A)$, then it is also of capacitary strong type $(m,\, A)$.*

2. Let A be any N-function. If J is of strong type $(A,\, A)$ and $0 < m < N$, then it is also of capacitary weak type (m, A).

7. Maximal operators, Lebesgue points and quasicontinuity in strongly nonlinear potential theory

We have shown that many maximal functions defined on some Orlicz spaces $\mathbf{L}_A$ are bounded operators on $\mathbf{L}_A$ if and only if they satisfy a capacitary weak inequality. We have shown also that (m, A)-quasievery x is a Lebesgue point for f in $\mathbf{L}_A$ sense and we have given an (m, A)-quasicontinuous representative for f when $\mathbf{L}_A$ is reflexive. See [35].

For $i, j \in N$, let $\theta_{i,j}$ be a complex valued function defined on $I\!R^N$ and such that $\theta_{i,j} \in \mathbf{L}_B$ for all N-functions B.

Let the sequence $(\theta_j)_j$ be such that

(1) $\theta_{i,j} * f \to \theta_j * f$ in $\mathbf{L}_B$ for all $f \in \mathbf{L}_B$
(2) $\theta_j * f_n \to \theta_j * f$ in $\mathbf{L}_B$ if $f_n \to f$ in $\mathbf{L}_B$.

Define the *maximal operator* $\mathcal{M}$

$$\mathcal{M}(f) = \sup_j |\theta_j * f|$$

and assume that $\mathcal{M}(f)$ is Lebesgue measurable on $I\!R^N$.

An operator $H : \mathbf{L}_A \to \mathbf{L}_A$ is of weak type (A,A) if

$$\forall f \in \mathbf{L}_A, \forall t > 0,\ \mathbf{m}(\{x : |H(f)(x)| > t\}) \leq \frac{1}{A\left(\frac{Ct}{|||f|||_A}\right)}$$

where C is a constant dependent only on A, and $\mathbf{m}$ is the Lebesgue measure on R^N. We say that H is of strong type (A,A) if

$$\forall f \in \mathbf{L}_A,\ |||H(f)|||_A \leq C|||f|||_A$$

where C is a constant dependent only on A.

Theorem 7.1. *1) Let A be an N-function and $\mathcal{M}$ the maximal operator defined by* (7)*. Suppose $\mathcal{M}$ is of strong type (A,A). Then*

$$\forall f \in \mathbf{L}_A, \forall t > 0, C_{k,A}\{x : \mathcal{M}(k * f)(x) > t\}) \leq A\left(C_A \frac{|||f|||_A}{t}\right).$$

C_A is the constant in the strong type.

If we suppose in addition that A verifies the Δ_2 condition, then there exists a constant C' dependent only on A, such that for all $t > 0$,

$$C_{k,A}(\{x : \mathcal{M}(k * f)(x) > t\}) \leq C' A\left(\frac{|||f|||_A}{t}\right).$$

2) Let A be an N-function satisfying the Δ_2 condition, and let $\mathcal{M}$ be the maximal operator defined by (7)*. Choose $k = G_m$ with $m > 0$. Let C be a constant dependent only on A and such that for all $t > 0$ and all $f \in \mathbf{L}_A$,*

$$C_{k,A}(\{x : \mathcal{M}(G_m * f)(x) > t\}) \leq CA\left(\frac{|||f|||_A}{t}\right).$$

Then $\mathcal{M}$ is of weak type (A, A).

If in addition we suppose that A^* verifies the Δ_2 condition, then $\mathcal{M}$ is of strong type (A, A).

3) Let A be an N-function and let $(k_i)_i$ be a sequence of positive integrable functions on $I\!R^N$ such that

a) $\int k_i(x)dx \to 1$, *as* $i \to \infty$
b) $\int_{\{|x| \geq \delta\}} k_i(x)dx \to 0$, *as* $i \to \infty$.

Then for any compact K in $I\!R^N$, $\lim_{i\to\infty} C_{k_i,A}(K) = A\left[\frac{1}{A^{-1}\left(\frac{1}{\mathbf{m}(K)}\right)}\right]$.

Theorem 7.2. *Let A be an N-function such that A and A^* satisfy the Δ_2 condition and let $\alpha = \alpha(A)$. Let m be a positive number and $f = G_m * g \in \mathbf{L}_{m,A}$, $0 < m\alpha < N$. Then $(m, A)-$quasievery x is a Lebesgue point for f in $\mathbf{L}_A-$sense, i.e.*

$$\lim_{r\to 0} \frac{1}{|B(x,r)|} \int_{B(x,r)} f(y)dy = \widetilde{f}(x) \text{ exists,}$$

and

$$\lim_{r\to 0} r^{\frac{-N}{\alpha}} |||f_x|||_{A,B(x,r)} = 0\,,$$

where f_x is defined as $f_x(y) = f(y) - \widetilde{f}(x)$.

Moreover, the convergence is uniform outside an open set of arbitrarily small $(m, A)-$capacity, $\widetilde{f}$ is an $(m, A)-$quasicontinuous representative for f, and

$$\widetilde{f}(x) = G_m * g \ (m, A) - q.e.$$

8. Wolff inequality in strongly nonlinear potential theory and applications

In this section we establish a Wolff type inequality for the strongly nonlinear potential theory. As applications, we give a relation between Bessel capacities and Hausdorff measure, and show that Riesz and Bessel capacities decrease under Lipschitz mapping in strongly nonlinear potential theory for reflexive Orlicz spaces. This generalizes the similar result in the nonlinear case and a result in the strongly one when the Lipschitz mapping is an orthogonal projection. See [39].

8.1. *A Wolff type inequality*

The strongly nonlinear potential associated to I_m and a positive measure μ is defined by

$$V^{\mu}_{I_m,A} = I_m * a^*(I_m * \mu).$$

The strongly nonlinear potential associated to G_m and μ is defined by

$$V^{\mu}_{G_m,A} = G_m * a^*(G_m * \mu).$$

We define $W^{\mu}_{m_1,A}(x) = \int\limits_0^1 t^{m-1} a^*(t^{m-N}\mu(B(x,t)))dt$, and

$W^{\mu}_{m_\infty,A}(x) = \int\limits_0^\infty t^{m-1} a^*(t^{m-N}\mu(B(x,t)))dt.$

Let $\mathcal{K}$ be a positive decreasing continuous function on $]0,\infty[$. For $x \in I\!R^N$, $x \neq 0$, define $\mathcal{K}(x) = \mathcal{K}(|x|)$.

We suppose in addition that $\mathcal{K}$ satisfies $\mathcal{K}(r) \leq L\mathcal{K}(2r)$ for some $L > 0$, and all sufficiently small $r > 0$. This implies that $L \geq 1$.

Special cases of such $\mathcal{K}$ are Bessel and Riesz kernels.

The strongly nonlinear potential associated to $\mathcal{K}$ and μ is defined by

$$V^{\mu}_{\mathcal{K},A} = \mathcal{K} * a^*(\mathcal{K} * \mu).$$

We set also $W^{\mu}_{\mathcal{K}_1,A}(x) = \int\limits_0^1 \mathcal{K}(t)t^{N-1} a^*(\mathcal{K}(t)\mu(B(x,t)))dt$, and

$W^{\mu}_{\mathcal{K}_\infty,A}(x) = \int\limits_0^\infty \mathcal{K}(t)t^{N-1} a^*(\mathcal{K}(t)\mu(B(x,t)))dt.$

Lemma 8.1. *Let A be an N-function such that A^* verifies the Δ_2 condition. Then there is a constant C such that for all positive measures μ,*

$$V^{\mu}_{\mathcal{K},A}(x) \geq C W^{\mu}_{\mathcal{K}_\infty,A}(x), \forall x \in I\!R^N.$$

The reverse inequality is false in general, but we establish an inequality in term of integrals for Riesz and Bessel kernels. Hence we obtain the following Wolff type inequality.

Theorem 8.1. *Let A be an N-function such that A and A^* satisfy the Δ_2 condition and let μ be a positive Radon measure. Let $0 < m < N$. There* *are constants C and C' such that*

$$C\int W^{\mu}_{m_\infty,A}d\mu \le \int V^{\mu}_{I_m,A}d\mu \le C'\int W^{\mu}_{m_\infty,A}d\mu$$

and

$$C\int W^{\mu}_{m_1,A}d\mu \le \int V^{\mu}_{G_m,A}d\mu \le C'\int W^{\mu}_{m_1,A}d\mu.$$

8.2. *Capacity and Hausdorff measure*

We have established the relation between Bessel capacities and Hausdorff measure and also a condition in term of Hausforff measure for a Bessel capacity of a set to be null. In this subsection we establish a converse deeper result that generalizes a theorem by V. P. Havin and V. G. Maz'ya [62, Theorem 7.1].

We begin by recalling some definitions about Hausforff measure.

Let $h : [0, +\infty[\to \overline{I\!R}$ be an increasing function satisfying $h(0) = 0$. For a subset $X \subset I\!R^N$ consider coverings of X by countable unions of (open or closed) balls $\{B(x_i, r_i\}_{i\ge1}$ with radii $\{r_i\}_{i\ge1}$. Let s be such that $0 < s \le \infty$, and define a set function $\Lambda_h^{(s)}$ by $\Lambda_h^{(s)}(X) = \inf \sum_{i\le1} h(r_i)$, where the infimum is taken over all such coverings with $r_i \le s$ for all $i \ge 1$.

Because $\Lambda_h^{(s)}(X)$ is a decreasing function of s, the *Hausdorff measure* of X with respect to the function h is defined by

$$\Lambda_h(X) = \lim_{s\to0} \Lambda_h^{(s)}(X).$$

The *Hausdorff content* or the *Hausdorff capacity* is the set function $\Lambda_h^{(\infty)}$ and we know that $\Lambda_h(X) = 0$ if and only if $\Lambda_h^{(\infty)}(X) = 0$.

Recall that if $m > J(A,N)$, there exists a positive constant $C = C(A, N, m)$ such that $B_{m,A}(X) \ge C$ for all set X such that $X \ne \emptyset$. Hence we must avoid this case when we work with sets with null capacity.

Theorem 8.2. *Let A be an N-function such that A and A^* satisfy the Δ_2 condition and X be a subset of $I\!R^N$. Let $0 < m \le J(A,N)$, and let h be an increasing function on $[0, +\infty[$ satisfying $h(0) = 0$, and $\int_0^1 t^{m-1}a^*(t^{m-N}h(t))dt < \infty$. Then*

$$\Lambda_h(X) = 0 \textit{ if } B_{m,A}(X) = 0.$$

8.3. *Lipschitz mappings and capacities*

We know that Bessel (and also Riesz) capacities in reflexive Orlicz spaces are non increasing under orthogonal projection of sets. Here we generalize this result to Lipschitz maps.

Theorem 8.3. *Let A be an N-function such that A and A^* satisfy the Δ_2 condition and let $0 < m < N$. Assume that $E \subset R^N$ and that $\Phi : E \to I\!R^N$ is a Lipschitz mapping, i.e. there is a constant L such that Φ satisfies*

$$|\Phi(x) - \Phi(y)| \leq L\,|x - y|$$

for all $x, y \in E$. Then there is a constant C depending only on L, m, N and A, such that

$$B_{m,A}(\Phi(E)) \leq C B_{m,A}(E)$$

and

$$R_{m,A}(\Phi(E)) \leq C R_{m,A}(E).$$

9. On the A-Laplacian

We establish, for Orlicz spaces $\mathbf{L}_A(I\!R^N)$ such that A satisfies the Δ_2 condition, the non resolubility of the A-Laplacian equation $\Delta_A u + h = 0$ on $I\!R^N$, and $\int h \neq 0$, if $I\!R^N$ is A-parabolic. For a large class of Orlicz spaces including Lebesgue spaces $\mathbf{L}^p$ $(p > 1)$, we prove also that the same equation, with any bounded measurable function h with compact support, has a solution with gradient in $\mathbf{L}_A(I\!R^N)$ if $I\!R^N$ is A-hyperbolic. See [37].

Definition 9.1. Let A be an N-function and K a compact set in $I\!R^N$. The A-capacity of K is defined by

$$\Gamma_A(K) = \inf\left\{|||\nabla u|||_A : u \in C_0^\infty(I\!R^N),\ u = 1 \text{ in a neighborhood of } K\right\}.$$

The space $I\!R^N$ is said to be A-parabolic if $\Gamma_A(K) = 0$ for all compact subsets $K \subset I\!R^N$ and A-hyperbolic otherwise.

The A-Dirichlet space $\mathbf{L}_A^1(I\!R^N)$ is the space of functions $u \in W_{A,loc}^1(I\!R^N)$ (i.e. u is locally in $W^1\mathbf{L}_A(I\!R^N)$) admitting a weak gradient such that $|||\nabla u|||_A < \infty$.

Let A be any N-function and let a be its derivative. The A-Laplacian of a function f on $I\!R^N$ is defined by $\Delta_A f = \operatorname{div}\Big(\frac{a(|\nabla f|)}{|\nabla f|}.\nabla f\Big)$.

A function $u \in W_{A,loc}^1(I\!R^N)$ is said to be a weak solution to the equation

$$\Delta_A u + h = 0$$

if for all $\varphi \in C_0^1(I\!R^N)$, we have

$$\int \langle \frac{a(|\nabla u|)}{|\nabla u|}.\nabla u, \nabla\varphi\rangle \, d\lambda = \int h\varphi \, d\lambda.$$

Let $D \subset I\!R^N$ be a non empty bounded domain. The Banach space $\mathcal{E}_A(D)$ is the space of functions $u \in W^1_{A,loc}(I\!R^N)$ such that

$$|||\nabla u|||_A^D := |||u|||_{A,D} + |||\nabla u|||_A < \infty.$$

We denote by $\mathcal{E}_A^0(D)$ the closure of $C_0^1(I\!R^N)$ in $\mathcal{E}_A(D)$.

9.1. *A non resolvability result*

Theorem 9.1. *Let A be an N-function satisfying the Δ_2 condition. Suppose that $I\!R^N$ is A-parabolic and let $h \in \mathbf{L}_1(I\!R^N)$ be such that $\int h \, d\lambda \neq 0$. Then the equation*

$$\Delta_A u + h = 0 \tag{2}$$

has no weak solution on $\mathbf{L}_A^1(I\!R^N)$.

Corollary 9.1. *Let $\mathbf{L}_A(I\!R^N)$ be a reflexive Orlicz space such that $\alpha^* \leq \frac{N}{N-1}$. Let $h \in \mathbf{L}_1(I\!R^N)$ be such that $\int h \, d\lambda \neq 0$. Then the equation (2) has no weak solution on $\mathbf{L}_A^1(I\!R^N)$.*

Remark 9.1. When $A(t) = p^{-1}t^p$, $\mathbf{L}_A = \mathbf{L}^p$ is the usual Lebesgue space and $\alpha^* = \frac{p}{p-1}$. Hence the condition $\alpha^* \leq \frac{N}{N-1}$ is exactly the condition $N \leq p$. Thus our result recovers the one in [58].

9.2. *A resolvability result*

In this subsection we resolve the equation $\Delta_A u + h = 0$ under some assumptions on the N-function A, and on the function h. We begin by the following type Poincaré inequalities for Orlicz-Sobolev functions. See [36].

Theorem 9.2. *1) Let A be an N-function such that A and A^* satisfy the Δ_2 condition. Let E be any measurable set in $I\!R^N$, such that $0 < \lambda(E) < \infty$. Then there exists a positive constant C such that*

$$|||u - u_E|||_{A,E} \leq C \, |||\nabla u|||_{A,E} \, ,$$

for all $u \in W^1_{A,loc}(I\!R^N)$, where $u_E = \frac{1}{\lambda(E)} \int\limits_E u \, d\lambda$ is the mean value of u on E.

2) Let A be an N-function such that A and A^* satisfy the Δ_2 condition.

Let E be any measurable set in $I\!\!R^N$, such that $0 < \lambda(E) < \infty$. Then there exists a positive constant C such that

$$\int_E |u - u_E| \, d\lambda \leq C \, |||\nabla u|||_{A,E} \, ,$$

for all $u \in W^1_{A,loc}(I\!\!R^N)$.
3) Let A be an N-function such that A and A^* satisfy the Δ_2 condition. Suppose that $I\!\!R^N$ is A-hyperbolic. Let E be any non empty bounded domain in $I\!\!R^N$. Then there exists a positive constant C such that for all $u \in \mathcal{E}^0_A(E)$

$$\int_E |u| \, d\lambda \leq C \, |||\nabla u|||_A \, .$$

Recall that for all $f \in \mathbf{L}_A$ such that $|||f|||_A > 1$, we have $\int A \circ f \, d\lambda > |||f|||_A$. We set $s(A) = \inf \left\{ \frac{\log \int A \circ f \, d\lambda}{\log |||f|||_A} - 1, \, f \in \mathbf{L}_A, \ |||f|||_A > 1 \right\}$. Hence $s(A) \geq 0$. One can consult [37] for some examples of N-functions satisfying $s(A) > 0$.

Now we are ready to solve the A-Laplace equation.

Theorem 9.3. *Let $\mathbf{L}_A$ be a reflexive Orlicz space such that $s(A) > 0$. Let $h \in \mathbf{L}^\infty(I\!\!R^N)$ have compact support. Then the equation $\Delta_A u + h = 0$ has a weak solution $u \in \mathbf{L}^1_A(I\!\!R^N)$ if $I\!\!R^N$ is A-hyperbolic.*

Remark 9.2. 1) We have in fact solved the equation in the space $\mathcal{E}^0_A(D) \subset \mathbf{L}^1_A(I\!\!R^N)$.

2) When $A(t) = p^{-1}t^p$, $p > 1$, $\mathbf{L}_A = \mathbf{L}^p$ is the usual Lebesgue space, we have $s(A) = p - 1 > 0$. Thus we recover the result in [94], when the manifold M is $I\!\!R^N$.

Corollary 9.2. *Let $\mathbf{L}_A(I\!\!R^N)$ be a reflexive Orlicz space such that $s(A) > 0$ and $\alpha < N$. Suppose that $h \in \mathbf{L}^\infty(I\!\!R^N)$ has compact support. Then the equation $\Delta_A u + h = 0$ has a weak solution $u \in \mathbf{L}^1_A(I\!\!R^N)$.*

References

1. D. R. Adams, *Traces of potentials arising from translation invariant operators*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. **25** (1971) 203-217.
2. D. R. Adams, *Maximal operators and capacity*, Proc. Amer. Math. Soc., **34** (1972) 152-156.
3. D. R. Adams, *Traces of potentials.* II, Indiana Univ. Math. J., **22** (1973) 907-918.
4. D. R. Adams, *A note on Riesz potentials*, Duke Math. J., **42** (1975) 765-778.

5. D. R. Adams, *On the existence of strong type estimates in* $\mathbb{R}^N$, Ark. Mat., **14** (1976) 125-140.
6. D. R. Adams, *Sets and functions of finite* L^p*-capacity*, Indiana Univ. Math. J., **27** (1978a) 611-627.
7. D. R. Adams, *Quasi-additivity and sets of finite* L^p*-capacity*, Pacific Journal Math., **79** (1978) 283-291.
8. D. R. Adams, *Capacity and the obstacle problem*, Appl. Math. Optim., **8** (1981) 39-57.
9. D. R. Adams, *Lectures on* L^p*-Potential Theory*, Department of Mathematics, University of Umeå, 1981.
10. D. R. Adams, *The exceptional sets associated with the Besov spaces*, in Linear and Complex Analysis Problem Book. 199 Research Problems (V. P. Havin, S. V. Hruščëv, N. K. Nikol'skii, eds.), Lecture Notes in Math. **1043**, 515-518, Springer, Berlin Heidelberg, 1984. Also in Linear and Complex Analysis Problem Book 3, Part II (V. P. Havin, N. K. Nikol'skii, eds.) Lecture Notes in Math. **1574**, 169-172, Springer, Berlin Heidelberg, 1994.
11. D. R. Adams, *Weighted nonlinear potential theory*, Trans. Amer. Math. Soc. **287** (1986) 73-94.
12. D. R. Adams, *A note on the Choquet integrals with respect to Hausdorff capacity*, in Function Spaces and Applications, Proc. Lund 1986 (M. Cwikel, J. Peetre, Y. Sagher, H. Wallin, eds.) Lecture Notes in Math. **1302**, 115-124, Springer, Berlin Heidelberg, 1988.
13. D. R. Adams, *A sharp inequality of J. Moser for higher order derivatives*, Ann. of Math., **128** (1988) 385-398.
14. D. R. Adams and A. Heard, *The necessity of Wiener test for some semi-linear elliptic equations*, Indiana Univ. Math. J., **41** (1992) 109-124.
15. D. R. Adams and L. I. Hedberg, *Inclusion relation among fine topologies in nonlinear potential theory*, Indiana Univ. Math. J., **33** (1984) 117-126.
16. D. R. Adams, L. I. Hedberg, *Function spaces and potential theory.* Springer-Verlag, Berlin, New York 1996.
17. D. R. Adams and J. L. Lewis, *Fine and quasi connectedness in nonlinear potential theory*, Ann. Inst. Fourier (Grenoble) **35:1** (1985) 57-73.
18. D. R. Adams and N. G. Meyers, *Thinness and Wiener criteria for nonlinear potentials*, Indiana Univ. Math. J., **22** (1972) 169-197.
19. D. R. Adams and N. G. Meyers, *Bessel potentials. Inclusion relations among classes of exceptional sets*, Indiana Univ. Math. J., **22** (1973) 873-905.
20. D. R. Adams and M. Pierre, *Capacitary strong type estimates in semi-linear problems*, Ann. Inst. Fourier (Grenoble) **41** (1991) 117-135.
21. D. R. Adams and J. C. Polking, *The equivalence of two definitions of capacity*, Proc. Amer. Math. Soc., **37** (1973) 529-534.
22. R. Adams, *Sobolev spaces,* Acad. Press, New York, London, Torento, (1975).
23. R. Adams, *On the Orlicz-Sobolev imbedding theorem*, J. Func. Analysis, **24** (1977) 241-257.
24. H. Aikawa, *Tangential boundary behavior of Green potentials and contractive properties of* L^p*-capacities*, Tokyo Math. J. **9** (1986) 223-245.
25. H. Aikawa, *Comparison of* L^p*-capacity and Hausdorff measure*, Complex

Variables, **15** (1990)223-232.

26. H. Aikawa, *Quasiadditivity of Riesz capacity*, Math. Scand., **69** (1991) 15-30.
27. H. Aikawa, *Quasiadditivity of capacity and minimal thinness*, Ann. Acad. Sci. Fenn. Ser. A. I. Math. **18** (1993) 65-75.
28. N. Aïssaoui, *Note sur la capacitabilité dans les espaces d'Orlicz*, Ann. Sci. Math. Québec. **19** (1995) 107-113.
29. N. Aïssaoui, *On the continuity of Bessel potentials in Orlicz spaces*, Collect. Math. **47**, 1 (1996) 77-89.
30. N. Aïssaoui, *Instability of capacity in Orlicz spaces*, Potential analysis, **6** (1997) 327-346.
31. N. Aïssaoui, *Bessel potentials in Orlicz spaces*, Revista Matemática de la Universidad Complutense de Madrid, Vol. **10**, $\mathbf{N^{O}.1}$ (1997) 55-79.
32. N. Aïssaoui, *Some developments of strongly nonlinear potential theory*, Libertas Mathematica, Vol.**XIX** (1999) 155-170.
33. N. Aïssaoui, *On the continuity of strongly nonlinear potential*, Math. Slovaca, **51** (2001), $n^{O}2$, 215-228.
34. N. Aïssaoui, *Capacitary type estimates in strongly nonlinear potential theory and applications*, Revista Matemática de la Universidad Complutense de Madrid, Vol. **XIV**, num.2 (2001) 347-370.
35. N. Aïssaoui, *Maximal operators, Lebesgue points and quasicontinuity in strongly nonlinear potential theory*, Acta Math. Univ. Comenianae, Vol **LXXI**, 1 (2002) 35-50.
36. N. Aïssaoui, *Strongly nonlinear potential theory on metric spaces*, Abstr. Appl. Anal., **7**, no. 7, (2002) 357-374.
37. N. Aïssaoui, *On the A-Laplacian*, Abstr. Appl. Anal., **2003:13** (2003) 743-755.
38. N. Aïssaoui, *Weighted strongly nonlinear potential theory*, Houston Journal of Mathematics, **29**, no. 1, (2003) 207-230.
39. N. Aïssaoui, *Wolff inequality in strongly nonlinear potential theory and applications*, African Diaspora Journal of Mathematics, Vol**1**, Number 1, (2004) 69-83.
40. N. Aïssaoui, *Another extension of Orlicz-Sobolev spaces to metric spaces*, Abstr. Appl. Anal., **2004:1**, (2004) 1-26.
41. N. Aïssaoui, *Orlicz-Sobolev spaces with zero boundary values on metric spaces*, Southwest Journal of Pure and Applied Mathematics, **1**, (2004) 10-32.
42. N. Aïssaoui, A. Benkirane, *Capacités dans les espaces d'Orlicz*, Ann. Sci. Math. Québec. **18** (1994) $n^{O}1$, 1-23.
43. N. Aïssaoui, A. Benkirane, *Potentiel non linéaire dans les espaces d'Orlicz*, Ann. Sci. Math. Québec. **18** (1994) $n^{O}2$, 105-118.
44. N. Aronszajn, F. Mulla, P. Szeptycki, *On spaces of potentials connected with L^p classes,* Ann. Inst. Fourier (Grenoble) **13** (1963) 211-306.
45. N. Aronszajn, K.T Smith, *Theory of Bessel potentials,* Ann. Inst. Fourier (Grenoble) **11** (1961) 383-475.
46. A. Benkirane, *Potentiel de Riesz et problèmes elliptiques dans les espaces d'Orlicz*, Thèse de Doctorat, U. L. B. (1988).
47. A. Benkirane and J. P. Gossez, *An approximation theorem in higher order*

Orlicz-Sobolev spaces and applications, Studia Math., **92** (1989) 231-255.

48. C. Fernström, *On the instability of capacity*, Ark. Mat., **15** (1977) 241-252.
49. B. Fugled, *On the theory of potentials in locally compact spaces*, Acta Math., **103** (1960) 139-215.
50. B. Fugled, *On generalized potentials of functions in the Lebesgue classes*, Mat. Scand., **8** (1960) 287-304.
51. B. Fugled, *Le théorème du minimax et la théorie fine du potentiel*, Ann. Inst. Fourier (Grenoble) **15:1** (1965) 65-88.
52. B. Fugled, *Quasi topology and fine topology*, Séminaire de Théorie du Potentiel (Brelot-Choquet-Deny) **10** (1965-1966) no. 12.
53. B. Fugled, *Applications du théorème minimax à l'étude de diverses capacités*, C. R. Acad. Sci. Paris Sér. A-B **266** (1968) 921-923.
54. B. Fugled, *The quasi topology associated with a countably subadditive set function*, Ann. Inst. Fourier (Grenoble) **21:1** (1971) 123–169.
55. B. Fugled, *Connexion en topologie fine et balayage des mesures*, Ann. Inst. Fourier (Grenoble) **21:3** (1971) 227–244.
56. B. Fugled, *Finely harmonic functions*, Lecture Notes in Math. **289**, Springer, Berlin Heidelberg, 1972.
57. B. Fugled, *A simple proof that certain capacities decrease under contraction*, Hiroshima Math. J. **19** (1989) 567-573.
58. V. Gol'dshtein, M. Troyanov, *Sur la non résolubilité du p-Laplacien sur* $I\!\!R^n$, C. R. Acad. Sci. Paris **326** (1998), 1185-1187.
59. K. Hansson, *Imbedding theorems of Sobolev type in potential theory*, Math. Scand., **45** (1979) 77-102.
60. V. P. Havin, *Non-linear potential theory*, Russian Math. Surveys **27:6** (1972) 71-148.
61. V. P. Havin, *Use of (p,l)-capacity in problems of the theory of exceptional sets*, Math. USSR-Sb., **19** (1973) 547-580.
62. V. P. Havin and V. G. Maz'ya, *Non-linear potential theory*, Russian Math.Surveys **27**:6 (1972) 71-148.
63. L. I. Hedberg, *Non-linear potentials and approximation in the mean by analytic functions*, Math. Z. **129** (1972) 299-319.
64. L. I. Hedberg, *On certain convolution inequalities*, Proc. Amer. Math. Soc., **36** (1972) 505-510.
65. L. I. Hedberg, *Bounded point evaluation and capacity*, J. Funct. Anal., **10** (1972) 269-280.
66. L. I. Hedberg, *Approximation in the mean by solutions of elleptic equations*, Duke Math. J., **40** (1973) 9-16.
67. L. I. Hedberg, *removable singularities and condenser capacities*, Ark. Mat., **12** (1974) 181-201.
68. L. I. Hedberg, *Two approximation problems in function spaces*, Ark. Mat., **16** (1978) 51-81.
69. L. I. Hedberg, *Nonlinear potential theory*, in Potential Theory, Proc. Int. Conf. Potential Theory, Nagoya 1990 (M. kishi, ed.) 43-54, de Gruyter, Berlin New York, 1991.
70. L. I. Hedberg and Th. H. Wolff, *Thin sets in nonlinear potential theory*, Ann.

Inst. Fourier (Grenoble) **33**:4 (1983) 161-187.
71. B. Jawerth, C. Pérez, G. Welland, *The positive cone in Triebel-Lizorkin spaces and the relations among potential and maximal operators.* Contemporary Mathematics, Proc. Conf. Boca Raton 1988 (M. Milman and T. Schonbek, Eds.) **107**, 71-971. Amer. Math. Soc., Providence, R. I, 1989.
72. M. A. Krasnosel'skii, Y. B. Rutickii, *Convex functions and Orlicz spaces,* P. Noordhoff, Groningen (The Netherlands), 1961.
73. A. Kufner, O. Hohn, S. Fučík, *Function spaces.* Academia, Prague, 1977.
74. W. A. J. Luxemburg, *Banach function spaces.* Thesis, Delft University, 1955.
75. V. G. Maz'ya, *On the theory of n-dimensional Schrodinger operator*, Izv. Akad. Nauk SSSR. Ser.Mat. **28** (1964) 1142-1172.
76. V. G. Maz'ya, *Polyharmonic capacity in the theory of the first boundary value problem*, Sibirsk. Mat. Zh. **6** (1965) 127-148.
77. V. G. Maz'ya, *On (p,l)-capacity, imbedding theorems and the spectrum of a self adjoint operators*, Mat. USSR-Izv. **7** (1973) 375-387.
78. V. G. Maz'ya, *Behaviour of solutions to the Dirichlet problem for the biharmonicoperator at a boundary point*, Lecture Notes in Math. **703** , 250-262, Springer, Berlin Heidelberg, 1979.
79. V. G. Maz'ya, *Sobolev spaces*, Springer-Verlag, Berlin-New York, 1985.
80. N. G. Meyers, *A theory of capacities for potentials of functions in Lebesgue classes*, Math. Scand., **26** (1970) 255-292.
81. N. G. Meyers, *Continuity of Bessel potentials*, Isr. J. Math. **11** (1972) 271-283.
82. N. G. Meyers, *Taylor expansion of Bessel potentials*, Indiana Univ. Math. J., **23** (1974) 1041-1049.
83. N. G. Meyers, *Continuity properties of potentials*, Duke Math. J., **42** (1975) 157-166.
84. N. G. Meyers, *Integral inequalities of Poincaré and Wirtinger type*, Arch. Rat. Mech. Anal., **68** (1978) 113-120.
85. B. Muckenhoupt and R.L. Wheeden, *Weighted norm inequalities for fractional integrals*, Trans. Amer. Math. Soc., **192** (1974) 261-274.
86. Yu. G. Reshetnyak, *On the concept of capacity in the theory of functions with generalized derivatives*, Sibirsk Mat. Z., **10** (1969) 1109-1138.
87. Yu. G. Reshetnyak, *On the boundary behavior of functions with generalized derivatives*, Sibirsk. Mat. Zh. **13** (1972), 411-419. English translation: Sibirian Math. J. **13** (1972), 285-290.
88. M. M. Rao, Z. D. Ren, *Theory of Orlicz spaces*, Dekker Inc, New York, 1991.
89. J. Serrin, *Local behavior of solutions of quasi-linear equations*, Acta Math., **111** (1964) 247-302.
90. J. Serrin, *Removable singularities of solutions of elliptic equations*, Arch. Rational Mech. Anal., **17** (1964) 65-78.
91. T. Sjödin, *Bessel potentials and extension of continuous functions on compact sets*, Ark. Mat., **13** (1975) 263-271.
92. E. M. Stein, *Singular integrals and differentiability properties of functions,* Princeton Univ. Press, Princeton, New Jersey 1970.
93. A. Torchinsky, *Interpolation of operators and Orlicz classes*, Studia Math.,

59 (1976) 177-207.
94. M. Troyanov, *Solving the p-Laplacian on manifolds*, Proc. Amer. Math. Soc. **128** (2000) n^{O}2, 541-545.
95. B. O. Turesson, *Nonlinear Potential Theory and Weighted Sobolev Spaces.* Lecture Notes in Mathematics 1736, Springer-Verlag, Berlin, Heidelberg, New York, 2000.
96. W. P. Ziemer, *Weakly differentiable functions*, Graduate Texts in Mathematics, Vol. 120, Springer-Verlag, New York, 1989.

Back on stochastic model for sandpile

Noureddine Igbida

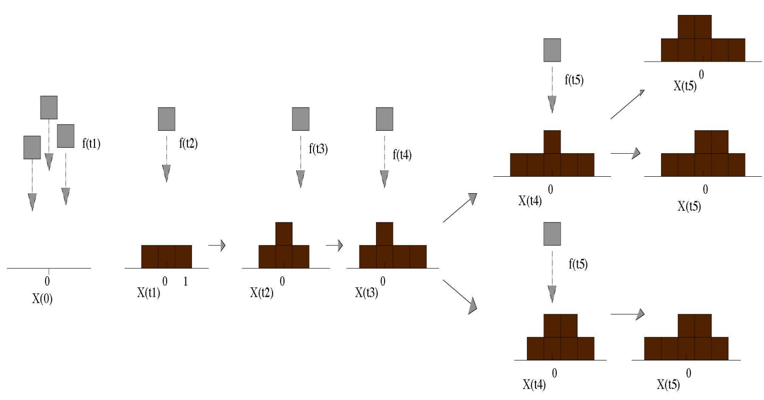
LAMFA, UMR 6140, Université de Picardie Jules Verne
33 rue Saint Leu, 80038 Amiens, France
E-mail: noureddine.igbida@u-picardie.fr

Our aim in this note is to give a simplified proof of the convergence of Evans-Rezakhanlou Stochastic Model to the evolution surfaces model of sandpile.

Keywords: Evans-Rezakhanlou Stochastic Model; Convergence.

1. Introduction

The stochastic model for sandpile was introduced by Evans and Rezakhanlou in[6] as a variant of physical models for sandpile. It corresponds to a Markov process $\Big(X(t)\Big)_{t\geq 0}$ defined by an infinitesimal generator describing the evolution of stack of unit cubes resting on the plane when new cubes are being added to the pile, by being placed either upon a heretofore unoccupied square in the plane or else upon the top of a current column.



At each time $t > 0$, the configuration $X(t)$ needs to be stable which means that the heights of any two adjacent columns of cubes can differ by at most one. So adding new cubes on existing pile, we ordain two possibilities to each cube:

- if the configuration is stable, then the cube remains in place
- otherwise, the cube has several downhill "staircases" along which it can move, and the cube will randomly select among the allowable downhill paths.

If we consider the case where the cubes can be also taken away, then a third possibility may be ordain

- the cube has several "upward staircases" along which it can move, and the cube will randomly select among the allowable upward paths.

Assuming that we continuously add cubes at random locations on a starting empty stack, Evans and Rezakhanlou studies the limit when one rascals in both space and time, so as to consider growing piles of more and more smaller and smaller cubes ? They prove that the macroscopic limit is rather simple and very connected Prigozhin model for sandpile (cf.[8] and[2]).

Since the work of Prigohzin (cf.[8] see also[2]), this has been well known (see also,[36] and the references therein) that the evolution of the surface of the sandpile when the angle of stability is equal to $\pi/4$ can be described by the following evolution problem

$$\begin{cases} \partial_t u + \partial \mathbb{I}_K(u) \ni f \\ \\ \hat{u}(0) = 0, \end{cases} \tag{1}$$

where $K = \Big\{ z \in W^{1,\infty}(\mathbb{R}^2) \cap L^2(\mathbb{R}^2) \ ; \ |\nabla z| \leq 1 \Big\}$, $\partial \mathbb{I}_K$ denotes the sub-differential operator of the indicator function (cf.[4]) of K and f describes a source term . Solution u is the height of the surface that grows up (resp. grows down) under sand addition (resp. sand removal) by a source called f. This is a critical slope model obtained by using the continuity equation and the gradient constraint $|\nabla u| \leq 1$ a.e. in $\mathbb{R}^2$ (see[8] and[2] for more details). Existence, uniqueness and numerical approximation the solution are well known by now for this model (cf.,[8],[25] and[7]). More precisely, for any $f \in BV_{loc}(0, T; L^2(\mathbb{R}^2))$, we know that that (1) has a unique solution u in the sense that $u \in W^{1,\infty}_{loc}(0, \infty; L^2(\mathbb{R}^2))$, $u(0) = 0$ and, for any $t \geq 0$, $u(., t) \in K$ and

$$\int_{\mathbb{R}^2} \Big(f(t,x) - \partial_t u(t,x) \Big) \Big(u(t,x) - \xi(x) \Big) \geq 0 \quad \text{for any } \xi \in K. \tag{2}$$

To give the connection between the two models, let N be a large integer. Assume the cubes are of side length $O(N^{-1})$ and cubes newly and randomly added at rate $O(N^{-1})$ are continuously falling downhill. For the description of the evolving of cubes, the authors of[6] introduce a probabilistic lattice model. They thus consider a Markov process for the height $X(t,i)$, defined for times $t \geq 0$ and sites $i \in \mathbb{Z}^2$. Resacaled source terms $f^+\Big(\frac{t}{N}, \frac{i}{N}\Big)$ and $f^-\Big(\frac{t}{N}, \frac{i}{N}\Big)$ control the rate new cubes are added to the pile or removed from it. Then, they proved that

$$\mathbb{E}\Big[\int_{\mathbb{R}^2} \Big|\frac{1}{N} X\Big(N\,t, [Nx]\Big) - u(t,x)\Big|^2\Big] \to 0 \quad \text{ as } N \to \infty, \tag{3}$$

where u is the unique solution of (1). As shown in,[6] the key estimates for the proof (3) are some kind of microscopic version of (2) (see (10)). To prove these estimates the authors of[6] prove some elementary intermediate estimates evolving various types of sets. Our aim in this note, is to improve these key estimates directly by using some simple arguments.

In the next section, we recall the stochastic model of Evans and Rezakhanlou. In Section 3, we prove the key estimates (10). Then, as in,[6] we introduce the discrete evolution problem associated with (1). Then, we give the proof of (3) by using some results of.[1]

2. The stochastic model for sandpile problem (cf.[6])

To describe the stochastic process for sandpile, we consider the lattice $\mathbb{Z}^n$. We equipped $\mathbb{Z}^n$ with an Euclidean norm and we say that $i, j \in \mathbb{Z}^n$ are adjacent, written $i \sim j$, provided

$$|i - j| \leq 1.$$

Without loose of generality, we restrict ourself to the cases $n = 2$, and we write $i = (i_1, i_2)$ to denote a typical site in $\mathbb{Z}^2$. Then, we introduce the Hilbert space

$$H := l^2(\mathbb{Z}^2) = \Big\{X \ : \ \mathbb{Z}^2 \to \mathbb{R}\ ;\ \|X\| := \sum_i X(i)^2 < \infty\Big\}.$$

A (stable) configuration is a mapping $X \ : \ \mathbb{Z}^2 \to \mathbb{Z}$ such that

$$\begin{cases} |X(i) - X(j)| \leq 1 & \text{if } i \sim j \\ \\ \text{and } X \text{ has bounded support.} \end{cases}$$

The state space is

$$S := \Big\{ X \ : \ \mathbb{Z}^2 \to \mathbb{Z} \ ; \ X \text{ is a configuration} \Big\},$$

and the set of stable configuration is

$$\hat{K} := \Big\{ X \in H \ ; \ |X(i) - X(j)| \leq 1 \text{ if } i \sim j \Big\}.$$

Let $\hat{f} \ : \ (0,\infty) \times \mathbb{Z}^2 \to \mathbb{Z}$ be a function, such that f^+ (resp. f^-) is controlling the rate new cubes are added (resp. removed) to the pile. The source of cubes $\hat{f}$ generates a stochastic process $(X(t), t \geq 0)$ in the state space S. It is clear that the probability that $X(t)$ be situated (at time t) in a given set Γ of E, under the condition that the movement of the system up to time s ($s < t$) is completely known, depends only on the state of the system at time s. In other words $(X(t), t \geq 0)$ is a Markov process. To study this process, we need to know its infinitesimal generator A, or more precisely $AF(X(t))$ for any $F \in B(S)$, the set of bounded functions. To this aim, let us consider $p^+(i,j,\xi)$ the probability that a cube placed on a given configuration $\xi \in S$ at the position i will end up at j after it has fallen downward over the stack with eight ξ. Likewise, let $p^-(i,j,\xi)$ be the probability that the removal of a cube from the pile at i will result in a removal at site j, after the cubes along a staircase each shifts downwards to fill in the gap created at site i. So, for any $i, \ j \ \in \mathbb{Z}^2$ we have

$$0 \leq p^{\pm}(i,j,\xi) \leq 1 \quad \text{ and } \quad \sum_{i \in \mathbb{Z}^2} p^{\pm}(i,j,\xi) = 1.$$

Thanks to,[6] we consider $c^{\pm}$ given by

$$c^{\pm}(j,X,t) = \sum_{i \in \mathbb{Z}^2} p^{\pm}(i,j,X(t))\, \hat{f}^{\pm}(t,i), \qquad \text{for any } (t,j) \in \mathbb{Z}^2 \times [0,\infty). \tag{4}$$

The parameter $c^+(j,X,\tau)$ (resp. $c^-(j,X,\tau)$) is highly nonlocal factor and records the rate, at time τ, new cubes come to rest at the site j after falling downhill (resp. the rate at which cubes are removed from the site j, at time τ). In particular, we have

$$\sum_{j \in \mathbb{Z}^2} c(j,X,t) = \sum_{i \in \mathbb{Z}^2} f(t,i).$$

Thanks again to,[6] the infinitesimal generator A of the Markov process

$(X(t), t \geq 0)$, is given by

$$A\,F(X(t)) = \sum_{j\in \mathbb{Z}^N} c^+(j, X, t)(F(X(t) + \delta_j) - F(X(t)))$$
$$- \sum_{j\in \mathbb{Z}^N} c^-(j, X, t)(F(X(t) - \delta_j) - F(X(t))) \quad \text{for any } t \geq 0, \tag{5}$$

where, for any $j \in \mathbb{Z}^2$, $\delta_j \; : \; \mathbb{Z}^2 \to \mathbb{N}$ is given by

$$\delta_j(i) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$$

In particular, we have

$$AF(X(t)) = (\mathcal{L}_t F)(X(t)) \quad \text{for any } (t, F) \in (0, \infty) \times B(S), \tag{6}$$

where $\mathcal{L}_t \; : \; F \in B(S) \to \mathcal{L}_t F \in L(S, \mathbb{R})$, for any $t \geq 0$, is the time dependent operator defined by

$$(\mathcal{L}_t F)(\xi) := \sum_{j\in \mathbb{Z}^N} c^+(j, \xi, t)(F(\xi + \delta_j) - F(\xi)) - \sum_{j\in \mathbb{Z}^N} c^-(j, \xi, t)(F(\xi - \delta_j) - F(\xi)). \tag{7}$$

3. Main results

Thanks to,[6] we know that the connection between the stochastic model and (1) is given through the following nonlinear dynamic in $l^2(\mathbb{Z}^2)$:

$$\begin{cases} \partial_t \hat{u} + \mathbb{I}_{\hat{K}}(\hat{u}) \ni \hat{f} & \text{for } t \geq 0 \\ \hat{u}(0) = 0, \end{cases} \tag{8}$$

where $\partial \mathbb{I}_{\hat{K}}$ denotes the sub-differential of $\mathbb{I}_{\hat{K}}$ in $l^2(\mathbb{Z}^2)$. Since $\hat{K}$ is closed and convex with $S \subseteq \hat{K}$, then (cf.[4]) for a given $\hat{f} \in BV(0, T; l^2(\mathbb{Z}^2))$, the evolution (8) has a unique solution $\hat{u} \in W^{1,\infty}_{loc}(0, \infty; H)$ such that $\hat{u}(0) = 0$ and, for any $t \geq 0$, $\hat{u}(., t) \in \hat{K}$ and

$$\sum_i \left(\hat{f}(t, i) - \partial_t \hat{u}(t, i)\right) \left(\hat{u}(t, i) - \hat{\xi}(i)\right) \geq 0 \quad \text{for any } \hat{\xi} \in \hat{K}.$$

We have

Theorem 3.1. *Assume that $\hat{f} \in BV(0, T; l^2(\mathbb{Z}^2))$. Let $\hat{u}$ be the solution of (8) and $(X(t), t \geq 0)$ be the stochastic process generated by $\hat{f}$. Then, we* *have*

$$\mathbb{E}\left[\sum_{i\in\mathbb{Z}^2}(X(t,i)-\hat{u}(t,i))^2\right]\le\int_0^t\sum_{j\in\mathbb{Z}^2}\hat{f}(j,s)\Big)\,ds,\qquad \textit{for any } t\ge 0. \quad (9)$$

Thanks to,[6] recall that the proof of Theorem 3.1 is based on the following estimates.

Lemma 3.1. *Under the assumptions of Theorem 3.1, for any* $w\in S$, *we have*

$$\pm\sum_{j\in\mathbb{Z}^2}c^{\pm}(j,X,t)\,(X(t,j)-w(j))\le\pm\sum_{i\in\mathbb{Z}^2}\hat{f}(t,i)\,(X(t,i)-w(i))\quad\forall w\in S. \tag{10}$$

The estimates (10) are some kind of microscopic version of (2). To prove (10) the authors of[6] prove some elementary intermediate estimates evolving various types of sets. In these notes, we use essentially the following remark and give a direct and short proof of this result. Then, the proof of Theorem 3.1 follows more or less the same step of.[6] Indeed, Lemma 3.1 gives the connection between $X(t)$ and (8). Then, we transform (8) into an evolution problem in $L^2(\mathbb{R}^2)$ governed by a sub-differential operator of the indicator function of the set of rescaled configurations (see (14)). This transformation allows us to give the connection between (1), (15) and the stochastic model.

Remark 1.

(1) For a given $\xi\in S$, if $p^+(i,j,\xi)>0$, then there exists at least one staircase $i_0=i\sim i_1\sim\ldots\sim i_m=j$, such that $\xi(i_p)=\xi(i_{p+1})+1$ for any $p=0,1,\ldots,m-1$. Let us denote this staircase by $\mathcal{C}^+(i,j)$; i.e.

$$\mathcal{C}^+(i,j)=[i_0,i_1,\ldots,i_{m-1},j],$$

and, for any $k\in\mathcal{C}^+(i,j)$, we denote by $\tilde{k}$, the adjacent side to k such that $u(k)=u(\tilde{k})+1$. It is clear that $\mathcal{C}(i,j)$ may not be unique, so (it is not essential) to simplify the presentation, let us consider the application:

$$\mathcal{C}^+\ :\ \mathbb{Z}^2\times\mathbb{Z}^2\to\mathcal{C}^+(i,j)\text{a staircase between } i \text{ and } j.$$

(2) For a given $\xi \in S$, if $p^-(i,j,\xi) > 0$, then there exists at least one upward staircase $i_0 = i \sim i_1 \sim ... \sim i_m = j$, such that $\xi(i_p) = \xi(i_{p+1}) + 1$ for any $p = 0, 1, \ldots, m-1$. Let us denote this staircase by $\mathcal{C}^-(i,j)$; i.e.

$$\mathcal{C}^-(i,j) = [i_0, i_1, ..., i_{m-1}, j],$$

and, for any $k \in \mathcal{C}(i,j)$, we denote by $\tilde{k}$, the adjacent side to k such that $u(k) = u(\tilde{k}) - 1$. It is clear that $\mathcal{C}(i,j)$ may not be unique, so (it is not essential) to simplify the presentation, let us consider the application :

$$\mathcal{C}^- \; : \; \mathbb{Z}^2 \times \mathbb{Z}^2 \to \mathcal{C}^-(i,j) \text{ a staircase between } i \text{ and } j.$$

Proof of Lemma 3.1: Thanks to (4), we have

$$\sum_{j \in \mathbb{Z}^2} c^+(j, X, t) \, (X(t,j) - w(j))$$

$$= \sum_{j,i \in \mathbb{Z}^2} p^+(i,j,X(t)) \, \hat{f}^+(t,i) \, (X(t,j) - w(j))$$

$$= \sum_{j,i \in \mathbb{Z}^2} p^+(i,j,X(t)) \, \hat{f}^+(t,i) \, (X(t,i) - w(i))$$

$$+ \sum_{j,i \in \mathbb{Z}^2} p^+(i,j,X(t)) \, \hat{f}^+(t,i) \Big((w(i) - w(j)) - (X(t,i) - X(t,j)) \Big)$$

$$= I_1 + I_2$$

Since $\displaystyle\sum_{j \in \mathbb{Z}^2} p^+(i,j,X(t)) = 1$, for any $(t,i) \in \mathbb{Z}^2 \times (0,\infty)$, then it is clear that

$$I_1 = \sum_{i \in \mathbb{Z}^2} \hat{f}^+(t,i) \, (X(t,i) - w(i)).$$

Let us prove that $I_2 \le 0$. Thanks to Remark 1, we have

$$I_2 = \sum_{j,i \in \mathbb{Z}^2} p^+(i,j,X(t)) \, \hat{f}^+(t,i) \sum_{k \in \mathcal{C}^+(i,j)} \Big((w(k) - w(\tilde{k})) - (X(t,k) - X(\tilde{k},t)) \Big)$$

$$\le \sum_{j,i \in \mathbb{Z}^2} p^+(i,j,X(t)) \, \hat{f}^+(t,i) \sum_{k \in \mathcal{C}^+(i,j)} \Big((w(k) - w(\tilde{k})) - 1 \Big)$$

$$\le 0,$$

where we used the fact that $|w(k) - w(\tilde{k})| \le 1$ (since $w \in K$ and $k \sim \tilde{k}$). The proof of

$$\sum_{j \in \mathbb{Z}^2} c^-(j, X, t) \, (X(t,j) - w(j)) \ge \sum_{i \in \mathbb{Z}^2} \hat{f}^-(t,i) \, (X(t,i) - w(i)),$$

follows in the same way, we let the details to the reader. ∎

To simplify the presentation of the proof of Theorem 3.1, we divide the proof into two steps that we present in the following Lemmas.

Lemma 3.2. *Under the assumptions of Theorem 3.1, we have*

(1) For any $w \in S$ *and* $t \geq 0$*, we have*

$$\frac{1}{2}\mathcal{L}_t\Big(\sum_{i\in\mathbb{Z}^2}\Big(X(t,i)-w(i)\Big)^2\Big) \leq \sum_{j\in\mathbb{Z}^2}\hat{f}(t,j)(X(t,j)-w(j))+\frac{1}{2}\sum_{j\in\mathbb{Z}^2} f(t,j).$$

(2) For any $w \in W^{1,\infty}(0,T;H)$ *such that* $w(t) \in S$*, for any* $t \in [0,T)$*, we have*

$$\frac{1}{2}\sum_{i\in\mathbb{Z}^2}(X(t,i)-w(t,i))^2 \leq \int_0^t \Big[\sum_{i\in\mathbb{Z}^2}\frac{\partial w}{\partial s}(i,s)\,(w(i,s)-X(i,s))$$

$$+\sum_{j\in\mathbb{Z}^2} f(j,s)\,(X(j,s)-w(j,s)) + \frac{1}{2}\sum_{j\in\mathbb{Z}^2} f(j,s)\Big] + \mathcal{M}(t)$$

where $\Big(\mathcal{M}(t)\Big)_{t\geq 0}$ *is a martingale satisfying*

$$\mathbb{E}(\mathcal{M}(t)) = 0 \quad \textit{for any } t \geq 0.$$

Proof:

(1) Let us denote

$$I = \frac{1}{2}\,\mathcal{L}_t\Big(\sum_{i\in\mathbb{Z}^2}\Big(X(t,i)-w(i)\Big)^2\Big)$$

By definition of $\mathcal{L}$, we have

$$I = \frac{1}{2}\sum_{j\in\mathbb{Z}^2} c(j,X,t)\Big(\sum_{i\in\mathbb{Z}^2}\Big(T_j(X(t))(i)-w(i)\Big)^2-\sum_{i\in\mathbb{Z}^2}\Big(X(t,i)-w(i)\Big)^2\Big).$$

So,

$$I = \frac{1}{2}\sum_{j\in\mathbb{Z}^2} c(j,X,t)\,\Big(\sum_{i\in\mathbb{Z}}(T_j(X)(i)-X(i))(T_j(X)(i)+X(i)-2\,u(i))\Big)$$

$$= \frac{1}{2}\sum_{j\in\mathbb{Z}^2} c(j,X,t)\,(2\,X(j)+1-2\,u(j))$$

$$= \sum_{j\in\mathbb{Z}^2} c(j,X,t)\,(X(j)-u(j)) + \frac{1}{2}\sum_{j\in\mathbb{Z}^2} c(j,X,t).$$

Thanks to (4), we deduce that

$$I = \sum_{j\in \mathbb{Z}^2} c(j, X, t)\,(X(j) - u(j)) + \frac{1}{2}\sum_{j\in \mathbb{Z}^2} f(t, j),$$

and, using Lemma 3.1, the first step of the lemma follows.

(2) As in,[6] we use the following stochastic integral equation : for any $F : S\times(0,\infty) \to \mathbb{R}$ Lipchitz continuous in t and $F(X(.,0),0) = 0$, we have

$$F(X(.,t),t) = \int_0^t \left(\frac{\partial F}{\partial s} + \mathcal{L}_s\, F\right)(X(.,s)) + \mathcal{M}(t). \tag{11}$$

Let F be given by

$$F(\xi,t) = \frac{1}{2}\sum_{i\in \mathbb{Z}^2} \Big(\xi(i) - w(t,i)\Big)^2, \qquad \text{for any } (\xi,t)\in S\times(0,T).$$

Then,

$$\frac{\partial F}{\partial s}(\xi,s) = -\sum_{i\in \mathbb{Z}^2} \frac{\partial w}{\partial s}(i,s)\Big(\xi(i) - w(i,s)\Big), \qquad \text{for any } (\xi,t)\in S\times(0,T),$$

and (11) implies that, for any $t\ge 0$,

$$\frac{1}{2}\sum_{i\in \mathbb{Z}^2} (X(t,i) - w(t,i))^2 = \int_0^t \Big(\sum_{i\in \mathbb{Z}^2} \frac{\partial w}{\partial s}(i,s)\,(w(i,s) - X(i,s)) + \mathcal{L}_s(F(X(.,s),s) + \mathcal{M}(t).$$

Then, by using the first step, the result follows. ∎

Proof of Theorem 3.1: Using the fact that $\hat{u}$ is a solution of (8) and $X(t)\in \hat{K}$, for any $t\ge 0$, we have

$$\sum_{i\in \mathbb{Z}^2} \frac{\partial \hat{u}}{\partial s}(i,s)(\hat{u}(i,s) - X(i,s)) + \sum_{j\in \mathbb{Z}^2} \hat{f}(j,s)(X(j,s) - \hat{u}(j,s)) \le 0 \qquad \text{for any } t\ge 0.$$

Then, using the second part of Lemma 3.2 with $\hat{u}$, we deduce (9). ∎

At last, let us come back to the continuous model of Prigozhin (1) and assume that

$$f\in BV_{loc}(0,\infty; L^2(\mathbb{R}^2)) \text{ and there exists } R>0 \text{ such that } \mathrm{spt}(f)\subseteq B(0,R), \tag{12}$$

here $B(0,R)$ denotes the a ball with center 0 and radius R. Thanks to,[4] we know that (1) has a unique solution u in the sense that $u \in W^{1,\infty}_{loc}(0,\infty;L^2(I\!R^2))$, $u(0)=0$ and, for any $t \geq 0$, $u(.,t) \in K$ and

$$\int_{I\!R^2} \Big(f(t,x) - \partial_t u(t,x)\Big) \Big(u(t,x) - \xi(x)\Big) \geq 0 \quad \text{for any } \xi \in K.$$

For the connection with the stochastic model, we rescale the source term $f\Big(\frac{t}{N}, \frac{x}{N}\Big)$. We set

$$\hat{f}(t,i) = f\Big(\frac{t}{N}, \frac{x}{N}\Big) \quad \text{for any } (t,i) \in [0,\infty) \times Z\!\!Z^2,$$

and we consider $(X(t))_{t\geq 0}$ the Markov processus generated by $\hat{f}$ as explained in the previous section.

Theorem 3.2. *Under the assumption (12), we have*

$$I\!E\left[\int_{I\!R^2} \left|u(t,x) - \frac{1}{N}\, X(N\,t,[N\,x])\right|^2\right] \to 0 \quad \textit{as } N \to 0. \tag{13}$$

Proof: Let $\hat{u}$ be the solution of (8) and consider

$$u_N(t,x) = \frac{1}{N}\hat{u}\Big(N\,t,[Nx]\Big) \quad \text{and} \quad f_N(t,x) = f\Big(t, \frac{[Nx]}{N}\Big)$$

for any $(t,x) \in [0,T) \times I\!R^2$. It is not difficult to see that $u_N \in W^{1,\infty}_{loc}((0,\infty);L^2(I\!R^2))$ and, for any $t \geq 0$, $u_N(.,t) \in K_N$, where

$$K_N = \Big\{z \in L^2(I\!R^2) \ ; \ |u(x) - u(y)| \leq \frac{1}{N} \text{ for } |x-y| \leq \frac{1}{N}\Big\}. \tag{14}$$

In addition, for any $\xi \in K_N$, we have

$$\int_{I\!R^2} \Big(f_N(t,x) - \partial_t u_N(t,x)\Big) \Big(u_N(t,x) - \xi(x)\Big)$$

$$= \tfrac{1}{N^3} \sum_i \Big(\hat{f}(Nt,i) - \partial_t \hat{u}(Nt,i)\Big) \Big(\hat{u}(Nt,i) - \hat{\xi}(i)\Big),$$

where

$$\hat{\xi}(i) = N^3 \int_{I_i} \xi(x)\, dx \quad \text{and} \quad I_i = \{z \in I\!R^2 \ ; \ [Nz] = i\}.$$

Now, since $\xi \in K_N$ and $\hat{u}$ is the solution of (8), then $\hat{\xi} \in \hat{K}$ and

$$\int_{I\!R^2} \Big(f_N(t,x) - \partial_t u_N(t,x)\Big) \Big(u_N(t,x) - \xi(x)\Big) \geq 0 \text{ for any } \xi \in K_N.$$

In other words u_N is the solution of the nonlinear dynamic

$$\begin{cases} \partial_t u_N + \partial I\!I_{K_N}(u_N) \ni \hat{f}_N \\ \\ \hat{u}_N(0) = 0. \end{cases} \tag{15}$$

Thanks to,[1] we know that

$$u_N \to u \quad \text{ in } \mathcal{C}([0,T);L^2(I\!R^2)). \tag{16}$$

Indeed, $\cap_{N\in I\!N} K_N = K$ so that, letting $N \to \infty$, we have $\partial I\!I_{K_N}$ converges in the sense of graph to $\partial I\!I_K$ in $L^\infty(I\!R^2)$. On the other hand, as $N \to \infty$,

$$f_N \to f \quad \text{ in } L^2_{loc}((0,\infty);L^2(I\!R^2)).$$

Then, by using classical perturbation result for nonlinear semigroup (cf.[4]), (16) follows. At last, thanks to Theorem 3.1, we have

$$\begin{aligned}
&I\!E\left[\int_{I\!R^2}\left(\tfrac{1}{N}X\Big(Nt,[Nx]\Big) - u_N(t,x)\right)^2 dx\right] \\
&= I\!E\left[\tfrac{1}{N^4}\ \sum_{i\in Z\!\!Z^2} |X\Big(Nt,i\Big) - N\ u_N(t,i)|^2\ dx\right] \\
&= I\!E\left[\tfrac{1}{N^4}\ \sum_{i\in Z\!\!Z^2} |X\Big(Nt,i\Big) - \hat{u}\Big(Nt,i\Big)|^2\ dx\right] \\
&\le \tfrac{1}{N^4}\ \int_0^{Nt} \sum_{i\in Z\!\!Z^2} \hat{f}\Big(s,i\Big)\ ds \\
&\le \tfrac{1}{N^2}\int_0^{Nt}\int_{I\!R^2} f\left(\tfrac{s}{N},\tfrac{[Nx]}{N}\right) dsdx \\
&\le \tfrac{1}{N}\int_0^{t}\int_{I\!R^2} f\left(\tau,\tfrac{[Nx]}{N}\right) d\tau dx \\
&\to 0 \quad \text{ as } N \to \infty.
\end{aligned}$$

so, by using (16), we deduce (13).

Acknowledgements

This work was partially supported by the French A.N.R. Grant JC05-41831 and by "FLUPARTI" project (supported and funded by "Conseil Régional de Picardie").

References

1. F. Andreu, J. M. Mazon, J. D. Rossi and J. Toledo, The limit as $p \to \infty$ in a nonlocal $p-$Laplacian evolution equation. A nonlocal approximation of a model for sandpiles. To appear in Cal. Var. Partial Diff. Equ.
2. G. Aronson, L. C. Evans and Y. Wu, Fast/Slow diffusion and growing sandpiles. *J. Differential Equations*, 131:304–335, 1996.
3. J. W. Barrett and L. Prigozhin, Dual formulation in Critical State Problems. *Interfaces and Free Boundaries*, **8**, 349-370, 2006.
4. H. Brézis, Opérateurs maximaux monotones et semigroups de contractions dans les espaces de Hilbert. (French). *North-Holland Mathematics Studies, No. 5. Notas de Matemàtica (50). North-Holland Publishing Co., Amsterdam-London; American Elsevier Publishing Co., Inc., New York, 1973.*
5. S. Dumont and N. Igbida, Back on a Dual Formulation for the Growing Sandpile Problem. *To appear in Eur. Journal Appl. Math.*.
6. L. C. Evans and F. Rezakhanlou, A stochastic model for sandpiles and its continum limit. *Comm. Math. Phys.,* 197 (1998), no. 2, 325-345.
7. N. Igbida, Evolution Monge-Kantorovich equation. *submitted.*
8. L. Prigozhin, Variational model of sandpile growth. *Euro. J. Appl. Math.* , **7**, 225-236, 1996

On radial solutions for Navier boundary eigenvalue problem with p-biharmonic operator

Siham El Habib* and Najib Tsouli†

University Mohamed I, Faculty of Sciences
Department of Mathematics and Computer
Oujda, Morocco
*E-mails: *s-elhabib@hotmail.com, †tsouli@hotmail.com*

This work deals with the existence of radial solutions for a nonlinear eigenvalue problem involving the p-biharmonic operator.

Keywords: p-biharmonic; Radial solutions.

1. Introduction

Let us consider the nonlinear eigenvalue problem

$$(P_p)\begin{cases}\Delta_p^2 u=\lambda|u|^{p-2}u \quad in\Omega,\\ u=\Delta u=0 \quad on\partial\Omega,\end{cases}$$

where $1<p<+\infty$, $\Delta_p^2 u:=\Delta(|\Delta u|^{p-2}\Delta u)$ is the operator of fourth order called p-biharmonic operator and $\Omega=B_1$ is the unit ball of $\mathbb{R}^N$, $N\geq 2$.

The spectrum of the p-biharmonic operator was recently studied by many authors (cf.,[2],[3],[4],[5],[6],[7],[8] and the references therein).

P. Drábeck and M.Ôtani[4] studied the spectrum of the problem (P_p) in a bounded and smooth domain Ω of $\mathbb{R}^N$ $(N\geq 1)$.
In one dimensional case and $\Omega=[0,1]$,J. Benedikt (cf.[2]) gave the spectrum of the p-biharmonic operator under Dirichlet and Neumann boundary conditions.

In his thesis, M. Talbi (cf.[6]) studied a variety of problems with p-biharmonic. He has included there many important techniques and tools concerning such problems.

A. El Khalil, S. Kellati and A. Touzani (cf.[5]) showed that the spectrum of the p-biharmonic operator with weight and with Dirichlet boundary conditions contains at least one non-decreasing sequence of positive eigenvalues.

Our study deals with the existence of radial solutions for the problem (P_p). We would like to mention some works which were of main importance to achieve this work: The first one is of F. De Thelin[9] where he showed that the problem

$$-\Delta_p u = \lambda |x|^\alpha |u|^{p-2} u, u \in W_0^{1,p}(B_1), u \geq 0, u \neq 0, \tag{1}$$

has at least one positive radial solution. The second one is of A.Anane[1] where he determined all the eigenvalues associated with a radial eigenfunction for the problem (1) and some properties of the eigenfunctions.

We show similar results as those of A.Anane.[1] We prove that the problem (P_p) has at least a sequence of eigenvalues associated with a radial eigenfunction, any eigenvalue $\lambda_n (n \geq 1)$ associated to a radial eigenfunction is given according to the first eigenvalue λ_1 and any radial eigenfunction associated to $\lambda_n (n \geq 1)$ has a finite number of components of the nodal set defined by $\{x \in \Omega : u(x) \neq 0\}$.

2. Preliminaries

In this section, we give some results and notations that will be used throughout this paper:
For $0 < R < 1$, we denote by:

- $B_R = \{x \in I\!R^N : |x| < R\}$,
- $C_R = \{x \in I\!R^N : R < |x| < 1\}$,
- $\lambda_{1,R}$ (resp. $\mu_{1,R}$) the first eigenvalue of the problem (P) in B_R (resp. C_R),
- φ_R (resp ψ_R) the eigenfunction associated with $\lambda_{1,R}$ (resp $\mu_{1,R}$),
- $\lambda_1 = \lambda_{1,1}$.

Definition 2.1. For any radial function u, we associate a real valued function u defined by $u(x) = \varphi(|x|)$ where $|x|$ is the Euclidian norm of x in $I\!R^N$.

The problem (P_p) is equivalent to the following problem (for more details see[4])

$$(P'_p) \begin{cases} \text{Find } \ (v, \lambda) \in (L^p(\Omega) \setminus \{0\}) \times I\!R_+^* \ \text{ such that:} \\ f'_1(v) = \lambda g'_1(v), \end{cases}$$

where

$$f_1(v) = \frac{1}{p} \|v\|_p^p, \ g_1(v) = \frac{1}{p} \|\Lambda v\|_p^p$$

and

$$f_1'(v) = N_p(v),\ g_1'(v) = \Lambda(N_p(\Lambda v))$$

N_p design the Nemytskii operator defined by

$$N_p(v)(x) = \begin{cases} |v(x)|^{p-2}v(x) & if \quad v(x) \neq 0 \\ 0 \quad if \quad v(x) = 0. \end{cases}$$

and Λ the inverse operator of $-\Delta : W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) \to L^p(\Omega)$.

In the following lemma we give some properties of the operator Λ (cf .[4]):

Lemma 2.1. *(i) (Continuity): There exists a constant $c_p > 0$ such that*

$$\|\Lambda f\|_{2,p} \leq c_p \|f\|_p$$

holds for all $p \in]1, +\infty[$ and $f \in L^p(\Omega)$.
(ii) (Continuity) Given $k \in I\!N^*$, there exists a constant $c_{p,k} > 0$ such that

$$\|\Lambda f\|_{W^{k+2,p}} \leq c_{p,k} \|f\|_{W^{k,p}}$$

holds for all $p \in]1, +\infty[$ and $f \in W^{k,p}(\Omega)$.
iii) (Symmetry) The equality

$$\int_\Omega \Lambda u \cdot v dx = \int_\Omega u \cdot \Lambda v dx$$

holds for all $u \in L^p(\Omega)$ and $v \in L^{p'}(\Omega)$ with $p \in]1, +\infty[$.
(iv) (Regularity) Given $f \in L^\infty(\Omega)$, we have $\Lambda f \in C^{1,\alpha}(\bar{\Omega})$ for all $\alpha \in]0, 1[$; moreover, there exists $c_\alpha > 0$ such that

$$\|\Lambda f\|_{C^{1,\alpha}} \leq c_\alpha \|f\|_\infty.$$

(v) (Regularity and Hopf-type maximum principle) Let $f \in C(\bar{\Omega})$ and $f \geq 0$ then $w = \Lambda f \in C^{1,\alpha}(\bar{\Omega})$, for all $\alpha \in]0, 1[$ and w satisfies: $w > 0$ in Ω, $\frac{\partial w}{\partial n} < 0$ on $\partial\Omega$.
(vi) (Order preserving property) Given $f, g \in L^p(\Omega)$ if $f \leq g$ in Ω, then $\Lambda f < \Lambda g$ in Ω.

We recall now the following result

Proposition 2.1. *Let X be a reflexive Banach space, M a weakly closed subset of X. Suppose $\phi : M \longrightarrow X$ is weakly lower semi-continuous on M then if ϕ is coercive on M, there exists $u_0 \in M$ such that*

$$\phi(u_0) = \inf_{v \in M} \phi(v)$$

see.[1]

We close this section by the following results (for more details see[4]):

Proposition 2.2. *If u is a nontrivial solution of problem (P_p) associated to an eigenvalue λ such that u does not change sign in Ω then $\lambda = \lambda_1$*

Proposition 2.3. *If u is a solution of the problem (P'_p) then $u \in C(\overline{\Omega})$*

Proposition 2.4. *If $f \in C(\overline{\Omega})$ and $f \geq 0$ then $\omega = \Lambda f \in C^{1,\alpha}(\overline{\Omega})$ for all $\alpha \in (0,1)$. ω satisfies $\omega > 0$ in Ω and $\frac{\partial \omega}{\partial n} < 0$ on $\partial\Omega$.*

3. Existence of radial solutions

The main objective of this section is to show that the problem (P_p) in B_R has an eigenvalue associated with a radial eigenfunction.

Proposition 3.1. *The problem (P_p) when $\Omega = B_R$ admits an eigenvalue λ_R associated with a radial positive eigenfunction.*

Proof. Let X be the completion of the space of radial functions of $I\!\!D(\Omega)$ under the norm of $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$.
Set

$$\lambda_R = \inf\{\|\Delta u\|_p^p;\ u \in X \ and \int_\Omega |u|^p dx = 1\},$$

$$f(u) = \frac{1}{p}\|\Delta u\|_p^p, \quad g(u) = \frac{1}{p}\int_\Omega |u|^p dx.$$

and

$$M = \{u \in X \ / \int_\Omega |u|^p dx = 1\}.$$

We verify first that f, X and M satisfies the conditions of the Proposition 2.1 then there exists $u \in X$, $u \neq 0$ such that $\forall v \in X$:

$$\int_\Omega \Delta(|\Delta u|^{p-2}\Delta u) v dx = \lambda_R \int_\Omega |u|^{p-2} u v dx. \tag{2}$$

By using Green formula, we obtain

$$-\int_\Omega \nabla(|\Delta u|^{p-2}\Delta u)\nabla v dx = \lambda_R \int_\Omega |u|^{p-2} u v dx \tag{3}$$

In spherical coordinates ($r = |x|$, $r \in [0,R]$ and $\theta \in S = [0,2\pi] \times ((-\frac{\pi}{2}, \frac{\pi}{2}))^{N-2}$) (3) becomes

$$-\int_0^R r^{N-1}(|\Delta\varphi|^{p-2}\Delta\varphi)'\phi' dr = \lambda_R \int_0^R |\varphi|^{p-2}\varphi\phi dr. \tag{4}$$

Where $u(x)=\varphi(r)$ and $v(x)=\phi(r)$, $\Delta u=\varphi''(r)+\frac{N-1}{r}\varphi'(r)$ and $\Delta\varphi(r)$ is a notation to design the quantity $\varphi''(r)+\frac{N-1}{r}\varphi'(r)$.
We multiply (4) by $\xi_1\in C^\infty(S)$ and after, we integrate on S. We obtain

$$-\int_\Omega \nabla(|\Delta u|^{p-2}\Delta u)\nabla v\ \xi dx=\lambda_R\int_\Omega |u|^{p-2}uv\ \xi dx, \tag{5}$$

where $\xi(x)=\xi_1(\theta)$. Since u is radial then

$$\nabla(|\Delta u|^{p-2}\Delta u)\nabla v\ \xi=\nabla(|\Delta u|^{p-2}\Delta u)\nabla(v\xi).$$

It follows that, for all $v\in I\!\!D(\Omega)$

$$\int_\Omega |\Delta u|^{p-2}\Delta u\Delta v dx=\lambda_R\int_\Omega |u|^{p-2}uv dx. \tag{6}$$

Then by density, we conclude that λ_R is an eigenvalue of the problem (P_p) in B_R associated with u which is a radial eigenfunction. □

Remark 3.1.
1/ The problem (P_p) is equivalent to the problem (P'_p) (see preliminaries section).
2/ If u is an eigenfunction of the problem (P_p) associated with λ_R then $v=-\Delta u$ is an eigenfunction associated with λ_R eigenvalue of (P'_p).
3/ $\lambda_R=\inf\{\|v\|_p^p; v\in L^p(\Omega); v\ radial\ and\ \int_\Omega|\Lambda v|^p=1\}$

Proposition 3.2. *The functionals φ_R and ψ_R are radial where φ_R (resp ψ_R) is an eigenfunction associated to the first eigenvalue $\lambda_{1,R}$ in B_R (resp. $\mu_{1,R}$ in the crown C_R).*

Proof. Let us show that φ_R is radial (a similarly proof is given for ψ_R). For this end, we prove that u does not change sign in Ω.
Since $v=-\Delta u$ is an eigenfunction of the problem (P'_p) associated with λ_R then $f_1(v)=\lambda_R g_1(v)$. From the p-homogeneity of f_1 and g_1 we deduce that

$$f_1(v)-\lambda_R g_1(v)=\inf_{u\in L^p(\Omega)\setminus\{0\},\ u\ radial} f_1(v)-\lambda_R g_1(v). \tag{7}$$

Then $f_1(v)-\lambda_R g_1(v)=0<f_1(|v|)-\lambda_R g_1(|v|)$. On the other hand, since $|\Lambda v|\leq\Lambda|v|$ then

$$f_1(v)-\lambda_R g_1(v)=0\geq f_1(|v|)-\lambda_R g_1(|v|)\Rightarrow f'_1(|v|)=\lambda_R g'_1(|v|)$$

i.e $N_p(|v|)=\Lambda(N_p(\Lambda|v|))$. It follows then that $|v|>0$, i.e., v does not change sign on Ω and consequently u does not change sign on Ω.
Proposition 2.2 allows us to conclude that $\lambda_R=\lambda_{1,R}$. However, $\lambda_{1,R}$ is

simple then there exists $t \in \mathbb{R}$ such that $\varphi_R = tu$ and Consequently φ_R is radial. □

Now, we are going to study the regularity of radial solutions. We associate to φ_R and ψ_R the real valued functions $\tilde{\varphi_R}$ and $\tilde{\psi_R}$ such that:

$$\varphi_R(x) = \tilde{\varphi_R}(r) \text{ and } \psi_R(x) = \tilde{\psi_R}(r) \text{ where } r = |x|.$$

Proposition 3.3. *(i)* $\tilde{\varphi_R} \in C^{1,\alpha}([0,R])$ *for some* $\alpha \in (0,1)$. *Moreover,* $\tilde{\varphi_R}'(0) = 0$, $\Delta\tilde{\varphi_R} < 0$ *in* $]0,R[$ *and* $\tilde{\varphi_R} > 0$ *in* $[0,R[$.
(ii) $\tilde{\psi_R} \in C^{1,\alpha}([R,1])$ *for some* $\alpha \in (0,1)$ *and there exists* $r_0 \in]R,1[$ *such that* $\tilde{\psi_R}'(r_0) = 0$. *Moreover,* $\Delta\tilde{\psi_R}(r) < 0$ *in* $]R,1[$.
(iii) The functionals $\tilde{\varphi_R}$ *and* $\tilde{\psi_R}$ *are at least of class* C^4 *in* $]0,R[$.

Proof. (i) Since φ_R does not change sign in B_R then $v_R = -\Delta\varphi_R$ is an eigenfunction of (P_p') associated with $\lambda_{1,R}$ which does not change sign in B_R. Suppose that $v_R \geq 0$ then by Propositions 2.3 and 2.4 (cf.[4]), we deduce that $v_R \in C(\overline{\Omega})$ then $\tilde{\varphi_R} \in C^{1,\alpha}([0,R])$ for some $\alpha \in (0,1)$, $\tilde{\varphi_R} > 0$ on $[0,R[$ and $\Delta\tilde{\varphi_R} < 0$ on $]0,R]$. Since $\tilde{\varphi_R}$ is radial then $\tilde{\varphi_R}'(0) = 0$.
(ii) Similarly to (i), we show that $\tilde{\psi_R} \in C^{1,\alpha}([R,1])$ for some $\alpha \in (0,1)$.
Since $\tilde{\psi_R}(R) = \tilde{\psi_R}(1) = 0$ then there exists $r_0 \in]R,1[$ such that $\tilde{\psi_R}'(r_0) = 0$.
Moreover, $\Delta\tilde{\psi_R} < 0$ in $[R,1]$.
(iii) Let us show that $\tilde{\varphi_R}$ is at least of class C^4 in $]0,R[$. By using the spherical coordinates, we have

$$(r^{N-1}(|\Delta\tilde{\varphi_R}|^{p-2}\Delta\tilde{\varphi_R})')' = \lambda_{1,R} r^{N-1}|\tilde{\varphi_R}|^{p-2}\tilde{\varphi_R} \tag{8}$$

where $\Delta\tilde{\varphi_R}(r)$ is a notation to design $\tilde{\varphi_R}''(r) + \frac{N-1}{r}\tilde{\varphi_R}'(r)$.
Since $\tilde{\varphi_R}$ is continuous then $(|\Delta\tilde{\varphi_R}|^{p-2}\Delta\tilde{\varphi_R})'$ is of class C^1 on $]0,R[$, i.e $\Delta\tilde{\varphi_R}$ is of class C^2 on $]0,R[$, then we deduce that $\tilde{\varphi_R} \in C^4(]0,R[)$. Similarly, we prove that $\tilde{\psi_R} \in C^4(]0,R[)$. □

4. Zeros points of eigenfunctions

First, We prove the following lemma where we state the relation between eigenvalues and eigenfunctions of the problem (P_p) in B_R and those in the unit ball B_1.

Lemma 4.1. *(i) For all* $R \in]0,1]$, *we have* $\lambda_{1,R} = R^{-2p}\lambda_1$ *and* $\varphi_R(x) = R^{\frac{2p-N}{p}}\varphi_1(\frac{x}{R})$ *for all* $x \in B_R$.
(ii) $\mu_{1,R}$ *is a real valued continuous function for all* $R \in]0,1[$, *strictly increasing and* $\lim \mu_{1,R} = +\infty$.

Proof. (i) Since φ_R is an eigenfunction of the problem (P_p) associated with $\lambda_{1,R}$ when $\Omega = B_R$ then

$$\int_\Omega |\Delta\varphi_R|^{p-2}\Delta\varphi_R\Delta v dx = \lambda_{1,R}\int_\Omega |\varphi_R|^{p-2}\varphi_R v dx \quad \forall v \in W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega). \tag{9}$$

Let us Consider the change of variable $x' = \frac{x}{R}$ and set $\varphi_R(x) = \varphi(x')$ and $v(x) = w(x')$ then

$$\int_{B_1} |\Delta\varphi|^{p-2}\Delta\varphi\Delta w dx' = \lambda_{1,R}R^{2p}\int_{B_1} |\varphi|^{p-2}\varphi w dx', \tag{10}$$

then $\lambda_{1,R}R^{2p}$ is an eigenvalue of problem (P_p) in B_1 associated with φ. Since φ is positive then $\lambda_1 = \lambda_{1,R}R^{2p}$ and $\lambda_{1,R} = R^{-2p}\lambda_1$.
Let us now show that $\varphi_R(x) = R^{\frac{2p-N}{p}}\varphi_1(\frac{x}{R})$. Indeed, we have

$$\int_{B_R} |\Delta\varphi_R|^p dx = \int_{B_1} |\Delta\varphi_1|^p dx. \tag{11}$$

Since λ_1 is simple then there exists $t > 0$ such that $\varphi = t\varphi_1$.
Hence

$$\int_{B_R} |\Delta\varphi_R|^p dx = R^{N-2p}t^p\int_{B_1} |\Delta\varphi_1|^p dx'.$$

It follows then that $t^p = R^{2p-N}$ Thus $\varphi_R(x) = R^{\frac{2p-N}{p}}\varphi_1(\frac{x}{R})$.
(ii) For the proof of this assertion, one can consult the thesis of A. Anane [1]. □

Now, we study the zero points of radial eigenfunctions.

Lemma 4.2. *If φ is a radial eigenfunction of the problem (P_p) then $N(\tilde\varphi) = \{r \in [0,1]$ such that $\tilde\varphi(r) = 0\}$ is finite and $0 \notin N(\tilde\varphi)$ where $\varphi(x) = \tilde\varphi(|x|)$.*

Proof. φ is an eigenfunction of (P_p) in B_1 then $\varphi(x) = 0$ for all x: $|x| = 1$. Since φ is radial then $1 \in N(\tilde\varphi)$, therefore $N(\tilde\varphi) \neq \emptyset$.
Set $r = \min\{N(\tilde\varphi) \setminus \{0\}\}$ since $\tilde\varphi$ does not change sign in $]0, r[$ then $0 \notin N(\tilde\varphi)$.
Let us show that $N(\tilde\varphi)$ is finite. Indeed, suppose by contrary that there exists a sequence $(r_n)_n \subset [0,1]$ such that $\tilde\varphi(r_n) = 0$. For a subsequence still denoted by $(r_n)_n$, we can assume that $\lim_n r_n = r$ for some $r \in [0,1]$, $r_n \neq r$, $\forall n \in \mathbb{N}$ then:

$$\tilde\varphi(r) = 0, \qquad \tilde\varphi'(r) = 0 \; and \qquad \tilde\varphi''(r) = 0.$$

Hence

$$\Delta\tilde{\varphi}(r) = \varphi''(r) + \frac{N-1}{r}\varphi'(r) = 0$$

which is in contradiction with (i) of Proposition 3.3. □

For $n \geq 1$, we define a set $Z(n)$ as follows :

$$Z(1) = \{R_1^1\} = \{1\},\ Z(n) = \{R_n^1, R_n^2, \dots, R_n^n\},$$

where

$$(\star\star)\begin{cases} R_n^n = 1 \\ R_n^i = a_{n-1}R_{n-1}^i \ if\ 1 \leq i \leq n-1\ and\ n \geq 2. \end{cases}$$

$(a_n)_n$ is a well defined sequence given by :

$$\begin{cases} a_1^{2p}\mu_{1,a_1} = \lambda_1 \\ a_n^{2p}\mu_{1,a_n} = \mu_{1,a_{n-1}}\ if\ n \geq 2. \end{cases}$$

We verify easily that:

$$0 < a_1 < a_2 < \ldots < a_n < \ldots < 1,$$

$$a_n^{2p}\mu_{1,a_n} = (\Pi_{i=1}^{n-1}a_i)^{-2p}\lambda_1,$$

$$0 < R_n^1 < \ldots < R_n^{n-1} = a_n < R_n^n = 1,$$

and

$$\begin{cases} R_n^i = a_i R_n^{i+1}\ if\ 1 \leq i \leq n-1 \\ R_n^1 = \Pi_{i=1}^{n-1}a_i = (\frac{\mu_{1,a_{n-1}}}{\lambda_1})^{-\frac{1}{2p}} \end{cases}$$

Lemma 4.3. *For any radial eigenfunction φ of the problem (P_p) in B_1 associated with an eigenvalue λ, we have*

$$N(\tilde{\varphi}) = Z(n)\ and\ \lambda = (R_n^1)^{-2p}\lambda_1$$

where $n = card(N(\tilde{\varphi}))$ and $\varphi(x) = \tilde{\varphi}(|x|)$.

Proof. We proceed by recurrence. For $n = 1$ we have $N(\tilde{\varphi}) = \{1\}$ and $\tilde{\varphi}$ does not change sign in $]0,1[$ then $\lambda = \lambda_1 = (R_1^1)^{2p}\lambda_1$.
Hypothesis: Suppose that for any radial eigenfunction u of (P_p) in B_1 such that $card(N(\tilde{u})) = n$ we have $N(\tilde{u}) = Z(n)$ and $\lambda = (R_n^1)^{-2p}\lambda_1$.
If ϕ is a radial eigenfunction of (P_p) in B_1 which satisfy $card(N(\tilde{\phi})) = n+1$ and $N(\tilde{\phi}) = \{T_1, T_2, \dots, Tn, T_{n+1} = 1\}$, we have

$$\begin{array}{l} \int_{B_1}|\Delta\phi|^{p-2}\Delta\phi\Delta v dx = \lambda\int_{B_1}|\phi|^{p-2}\phi v dx \\ \forall v \in W^{2,p}(B_1)\cap W_0^{1,p}(B_1). \end{array} \qquad (12)$$

Then by considering the variable change $x' = xT_n$, one obtains

$$\int_{B_{T_n}} |\Delta\omega|^{p-2}\Delta\omega\Delta v dx' = \lambda T_n^{2p}\int_{B_{T_n}} |\omega|^{p-2}\omega v dx \qquad \forall v \in W^{2,p}(B_{T_n}) \cap W_0^{1,p}(B_{T_n}), \tag{13}$$

where ω is the restriction of ϕ to B_{T_n} and $\omega(x') = \phi(x)$ then λT_n^{2p} is an eigenvalue of (P_p) associated to ω in B_{T_n}. From the precedent hypothesis, we get $\lambda T_n^{2p} = (R_n^1)^{-2p}\lambda_1$.
Similarly, the restriction of ϕ to C_{T_n} is an eigenfunction of constant sign on C_{T_n} associated with λ then $\mu_{1,T_n} = \lambda$ and $\mu_{1,T_n}T_n^{2p} = (R_n^1)^{-2p}\lambda_1$. We conclude that $T_n = a_n$, $T_{n-1} = a_{n-1} = R_n^{n-1} = a_{n-1}R_n^n$. Consequently, $N(\tilde{\phi}) = \{R_{n+1}^1, \ldots, R_{n+1}^{n+1}\} = Z(n+1)$. □

5. Main result

Set $\tilde{\varphi_1}(|x|) = \varphi_1(x)$ and $\tilde{\psi_{a_i}}(|x|) = \psi_{a_i}(x)$ for all $x \in B_1$.
It's easy to see that $\tilde{\varphi_1'}(1) \neq 0$ and $\tilde{\psi_{a_i}'}(1) \neq 0$. Set

$$u_n(x) = \begin{cases} \varphi_1(\frac{x}{R_n^1}) & if \quad |x| \leq R_n^1 \\ \delta_i\psi_{a_i}(\frac{x}{R_n^{i+1}}) & if \quad R_n^i \leq |x| \leq R_n^{i+1},\ i < n \ and \ n \geq 2 \end{cases}$$

where δ_i is defined by

$$\begin{cases} \delta_1 = \frac{\tilde{\varphi_1'}(1)}{a_1\tilde{\psi}'_{a_1}(a_1)} \\ \delta_i = \delta_1 \times \Pi_{j=2}^{i}(\frac{\tilde{\psi}'_{a_{j-1}}(1)}{a_j\tilde{\psi}'_{a_j}(a_j)}) \ if \ i \neq 1. \end{cases}$$

Let us consider the sequence $(\lambda_n)_n$ defined by $\lambda_n = (R_n^1)^{-2p}\lambda_1$.

Theorem 5.1. ***(i)*** *For all $n \geq 1$, u_n is a radial eigenfunction of problem (P_p) in B_1 associated to the eigenvalue λ_n.*
(ii) *If u is a radial eigenfunction of problem (P_p) in B_1 associated with an eigenvalue λ then there exists $n \geq 1$ and $t \in I\!R$ such that: $\lambda = \lambda_n$ and $u = tu_n$.*

Proof. (i) If $n = 1$ then $\lambda_n = \lambda_1$ and $u_n = \varphi_1$. The result holds.
If $n \geq 2$. Let v be the restriction of u_n to $B_{R_n^1}$ and w_i the restriction of u_n to the crown $C_{R_n^{i+1}}^{R_n^i} = \{x \in I\!R^N : R_n^i < |x| < R_n^{i+1}\}$.
Firstly, since $v(x) = \varphi_1(\frac{x}{R_n^1})$ then by lemma 4.1, v is an eigenfunction of (P_p) in $\Omega = B_{R_n^1}$ associated with $\lambda_{1,R_n^1} = (R_n^1)^{-2p}\lambda_1 = \lambda_n$.

Secondly, since ψ_{a_i} is an eigenfunction of (P_p) in $\Omega = C_{a_i} = \{x \in I\!R^N : a_i < |x| < 1\}$ associated to $\mu_{1,ai}$ then

$$\int_{C_{a_i}} |\Delta\psi_{a_i}|^{p-2}\Delta\psi_{a_i}\Delta v dx = \mu_{1,ai}\int_{C_{a_i}} |\psi_{a_i}|^{p-2}\psi_{a_i} v dx \quad \forall v \in W^{2,p}(C_{a_i}) \cap W_0^{1,p}(C_{a_i}). \tag{14}$$

Set $x' = R_n^{i+1}x$ and $w_i(x') = \psi_{a_i}(x)$ then we obtain

$$\int_{C_{R_n^{i+1}}^{R_n^i}} |\Delta w_i|^{p-2}\Delta w_i \Delta v dx' = (R_n^{i+1})^{-2p}\mu_{1,ai}\int_{C_{R_n^{i+1}}^{R_n^i}} |w_i|^{p-2} w_i v dx' \quad \forall v \in W^{2,p}(C_{R_n^{i+1}}^{R_n^i}) \cap W_0^{1,p}(C_{R_n^{i+1}}^{R_n^i}). \tag{15}$$

This shows that w_i is an eigenfunction of (P_p) in $C_{R_n^{i+1}}^{R_n^i}$ associated with $(R_n^{i+1})^{-2p}\mu_{1,ai}$. However, since $R_n^{i+1} = \frac{1}{a_i}R_n^i = (\Pi_{j=1}^i a_j)^{-1}R_n^1$ then $(R_n^{i+1})^{-2p} = (R_n^1)^{-2p}(\Pi_{j=1}^i a_j)^{2p} = (R_n^1)^{-2p}\frac{\lambda_1}{\mu_{1,ai}}$. From where , we conclude that w_i is an eigenfunction on (P_p) in $C_{R_n^{i+1}}^{R_n^i}$ associated with λ_n.
To complete the proof, we need to study the regularity of u_n in the neighborhood of the points x which verify $|x| = R_n^i$. We prove that u_n is of class C^4 at x: $|x| = R_n^i$.

Set $\tilde{v}(|x|) = v(x)$ and $\tilde{w_i}(|x|) = w_i(x)$. Since $\tilde{v}(R_n^1) = \tilde{w_i}(R_n^i) = 0$ $\forall 1 \le i \le n-1$ then u_n is continuous at x: $|x| = R_n^i$. The function u_n is of class C^1 at x : $|x| = R_n^i$. Indeed,
$\tilde{w_i'}(R_n^i) = \frac{\delta_i}{R_n^{i+1}}\tilde{\psi'_{a_i}}(a_i)$ and $\tilde{w'_{i-1}}(R_n^i) = \frac{\delta_i}{R_n^{i+1}}\tilde{\psi'_{a_i}}(1)$ then $\frac{\tilde{w'_{i-1}}(R_n^i)}{\tilde{w_i'}(R_n^i)} = 1$ i.e $\tilde{w'_{i-1}}(R_n^i) = \tilde{w_i'}(R_n^i)$.
We can also easily verify that $\tilde{v_1'}(R_n^1) = \tilde{w_1'}(R_n^1)$. The result then is proved.
We have

$$\Delta\tilde{w}_i(R_n^i) = \tilde{w_i''}(R_n^i) + \frac{N-1}{R_n^i}\tilde{w_i'}(R_n^i) \tag{16}$$

$$= 0 \tag{17}$$

$$= \Delta\tilde{w_{i-1}}(R_n^{i+1}) \tag{18}$$

then $\tilde{w_i''}(R_n^i) = -\frac{N-1}{R_n^i}\tilde{w_i'}(R_n^i)$ and $\tilde{w''_{i-1}}(R_n^i) = -\frac{N-1}{R_n^i}\tilde{w'_{i-1}}(R_n^i)$ what implies that $\tilde{w_i''}(R_n^i) = \tilde{w''_{i-1}}(R_n^i)$.
Similarly, we prove that $\tilde{v_i''}(R_n^1) = \tilde{w''_{i-1}}(R_n^1)$. It follows then that u_n is of class C^2 at x : $|x| = R_n^i$.
u_n is of class C^3 at x : $|x| = R_n^i$:
$\tilde{w}_i$ and $\tilde{v}$ verifies the equations:

$$r^{N-1}(|\Delta\tilde{w}_i(r)|^{p-2}\Delta\tilde{w}_i(r))' = \lambda_n\int_0^r s^{N-1}|\tilde{w}_i|^{p-2}\tilde{w}_i ds$$

$$r^{N-1}(|\Delta\tilde{v}(r)|^{p-2}\Delta\tilde{v}(r))' = \lambda_n \int_0^r s^{N-1}|\tilde{v}|^{p-2}\tilde{v}ds$$

then

$$r^{N-1}(p-1)(\Delta\tilde{w}_i)' = \lambda_n|\Delta\tilde{w}_i(r)|^{2-p}\int_0^r s^{N-1}|\tilde{w}_i|^{p-2}\tilde{w}_i ds$$

$$r^{N-1}(p-1)(\Delta\tilde{v})' = \lambda_n|\Delta\tilde{v}(r)|^{2-p}\int_0^r s^{N-1}|\tilde{v}|^{p-2}\tilde{v}ds$$

1^{st} **case**: if $1 < p < 2$ then at x : $|x| = R_n^1$, we have $(\Delta\tilde{v})'|_{r=R_n^1} = 0$ this implies that

$$\tilde{v}^{(3)}(R_n^1) + \frac{N-1}{R_n^1}\tilde{v''}(R_n^1) - \frac{N-1}{R_n^1}\tilde{v'}(R_n^1) = 0$$

However, $\tilde{v''}(R_n^1) = -\frac{N-1}{R_n^1}\tilde{v'}(R_n^1)$ then $\tilde{v}^{(3)}(R_n^1) = -\frac{N}{R_n^1}\tilde{v''}(R_n^1)$.
Since $\tilde{v''}$ is continuous at R_n^1 then $\tilde{v}^{(3)}$ is continuous at R_n^1 . The same proof is given for $\tilde{w}_i$.
2^{nd} **case**: if $p > 2$. By Dràbeck and Ôtani (cf.[4]), if $u(p)$ is a solution of the problem (P_p) associated to the eigenvalue $\lambda(p)$ then $\lambda(p')$ defined by $(\lambda(p'))^{\frac{1}{p'}} = (\lambda(p))^{\frac{1}{p}}$ is an eigenvalue of the problem $(P_{p'})$ associated with the eigenfunction $u(p')$ defined by $u(p') = \frac{1}{\lambda(p)}|\Delta u(p)|^{p-2}\Delta u(p)$ where $p' = \frac{p}{p-1}$.
We have $1 < p' < 2$ then similarly to the first case, we obtain $\tilde{v}^{(3)}(p')$ is continuous at R_n^1. Since $\tilde{v}(p') = \frac{1}{\lambda(p)}|\Delta\tilde{v}|^{p-2}\Delta\tilde{v}$, we deduce that $\tilde{v}^{(3)}(p)$ is continuous at R_n^1. Consequently, u_n is of class C^3 at x: $|x| = R_n^i$.
u_n is of class C^4 at x: $|x| = R_n^i$ is proved similarly as for the assertion: " u_n is of class C^3 ". The proof then is completed.
(ii) To prove this assertion, let u be a radial eigenfunction of (P_p) in B_1 associated to an eigenvalue λ then by the preceding results we have $\lambda = \lambda_1(R_n^1)^{-2p} = \lambda_n$ where $n = card(N(\tilde{u}))$. The restriction v (resp. w_i) of u to $B_{R_n^1}$ (resp. $C_{R_n^{i+1}}^{R_n^i}$) is an eigenfunction of (P_p) in $B_{R_n^1}$ (resp. $C_{R_n^{i+1}}^{R_n^i}$) which does not change its sign in $B_{R_n^1}$ (resp. $C_{R_n^{i+1}}^{R_n^i}$).
Since λ is simple and from (i) , we deduce that: $\lambda = \lambda_n$ and there exists $t \in I\!\!R$ such that $u = tu_n$. □

References

1. A. Anane, *Etude des valeurs propres et de la résonance pour l'opérateur p-Laplacien*, thèse de Doctorat, (Université Libre de Bruxelle, 1988).

2. J. Benedikt, *On the Discreteness of the Dirichlet and Neumann p-Biharmonic problem*, (Abst. Appl. Anal., Vol. 293(2004)), No 9, pp. 777-792.
3. J. Benedikt, *Uniqueness theorem for the p-biharmonic Equations*, (Electron. J. Differential Equations 2002), No 53, pp. 1-17.
4. P. Dràbek and M. Ôtani, *Global Bifurcation Result for the p-Biharmonic Operator*, (Electronic Journal Of Differential Equations 2001), No 48, pp. 1-19.
5. A. El Khalil, S. Kellati and A. Touzani, *On the spectrum of the p-Biharmonic Operator*, 2002-Fez Conference On partial differential Equations, (Electronic Journal of Differential Equations, Conference 09, 2002), pp. 161-170
6. M. Talbi, *Etude des problèmes elliptiques d'ordre quatre gouvernées par l' opérateur p-biharmonique*, Thèse de doctorat, (Faculté des sciences, Oujda, Maroc, 2007).
7. M. Talbi and N. Tsouli, *Existence and uniqueness of a positive solution for a non homogeneous problem of fourth order with weight*, 2005- Oujda International Conference on Nonlinear Analysis, (Electronic Journal of Differential Equations, Conference 14, 2006), pp. 231-240.
8. M. Talbi and N. Tsouli, *On the Spectrum of the weighted p-Biharmonic Operator with weight*, (Mediterr. J. Math. 4 (2007)).
9. F. De Thélin, *Quelques résultats d'existence et non existence pour une E.D.P elliptique non lineaire*, (C. R. Acad. Sc. Paris 1984), N 18 pp. 911-914.

Floquet states of periodically time-dependent harmonic oscillators

Y. Achkar, S. Sayouri, and A. L. Marrakchi

Laboratoire de Physique Théorique et Appliquée (LPTA)
Département de Physique
Faculté des Sciences Dhar-Mehraz
B.P. 1796, Fés-Atlas, Fés, Morocco

Floquet theory together with the resonating averages method (RAM) were used to solve the Schrödinger equation of periodically-time-dependent harmonic oscillators. The approach gives another way of solving this equation, and lead to identical analytical solutions to those published in the literature based on different other methods.

Keywords: Time-dependent harmonic oscillator; Floquet operators; Floquet states; Jump operators; Resonating averages method; Uncertainty relation.

1. Introduction

Numerous physical and mathematical studies have been devoted to the construction of consistent formalism of explicitly time-dependent quantum systems[1]-.[9]Among these quantum systems, the time-dependent harmonic oscillator has attracted considerable interest in the past few decades[3]-,[1218]-,[25] not only because it is a system that can be exactly solved and is considered as a great pedagogical tool, but also because it is a very relevant system and possessing many applications in different areas of physics: quantum optics, plasma physics, quantum field theory, gravitation, cosmology, etc.

The harmonic oscillator with a time-dependent mass and a constant frequency has been the subject of many researches[16]-,[1921]-.[25] Abdalla et al.[18] have studied this system in order to describe the electromagnetic field intensities in a Fabry-Perot cavity. The harmonic oscillator with a constant mass and a time-dependent frequency was used by Lemos et al.[20] in a theoretical approach of the expansion of the universe and by Paul[23] for the discovery of electromagnetic traps for charged and neutral particles (known as Paul-traps). In previous work,[24] we have introduced a direct theoretical approach for solving the equations of motion of the time-dependent harmonic oscillator, in Heisenberg picture, with some direct applications.

The purpose of this paper is to solve the Schrödinger evolution equation of periodically-time-dependent harmonic oscillator. We have established a theoretical approach based on the Floquet theory[1] combined with the resonating averages method elaborated by Lochak and Thiounn.[2] Hence,by using the Floquet decomposition operators and developing the evolution operator with the help of the resonating averages method from first order to second ameliorated order, we have determinate the Floquet operators, and Floquet states. Moreover, we have proposed a theoretical form of jump operators between the instantaneous Floquet states, and verified the uncertainty principle.

This paper is organized as follows: in the first part, we present the theoretical formalism of the purposed approach. In the second part, we apply the method for the forced harmonic oscillator, the harmonic oscillator with a time-dependent mass and a constant frequency, the harmonic oscillator with a constant mass and a time-dependent frequency, and the harmonic oscillator with a time-dependent mass and frequency. Some direct comparisons of our results with some published works are given,[3],[59]-,[1218]

2. Formalism

We consider a quantum system described by a time-dependent Hamiltonian such as

$$H(t) = H_0 + \mu H_1(t) \tag{1}$$

where H_0 is the Hamiltonian of the unperturbed oscillator, and $H_1(t)$ the Hamiltonian of perturbation with amplitude μ.

2.1. *Floquet approach*

In the case of periodically time-varying Hamiltonian, the Floquet theorem asserts the existence of an operator $V(t)$, a fundamental solution of the well-known Schrödinger evolution equation, in the form[1]

$$V(t) = T(t)e^{-iRt/\hbar} \tag{2}$$

Where $T(t)$ is a periodic unitary operator of the same period as $H_1(t)$, and R is a constant hermitian operator. The time-evolution operator can be written as

$$U(t,t_0) = V(t)V^{-1}(t_0) \tag{3}$$

in the interaction picture it satisfies the following differential equation

$$i\hbar\frac{dU_I(t,t_0)}{dt} = \mu H_I(t)U_I(t,t_0) \tag{4}$$

where

$$H_I(t) = e^{iH_0 t/\hbar} \tag{5}$$

According to the Floquet theorem, the Floquet states, solutions of the Schrödinger equation, are defined as

$$|\psi_n(t)\rangle = T(t)|\phi_n(t)\rangle \tag{6}$$

where $|\phi_n(t)\rangle$ are the eigenstates of R, with the eigenvalues ε_n, such as

$$i\hbar\frac{d|\Phi_n(t)\rangle}{dt} = R|\Phi_n(t)\rangle \tag{7}$$

and

$$|\phi_n(t)\rangle = e^{-i\varepsilon_n t/\hbar}|n\rangle \tag{8}$$

$|n\rangle$ are the number states of the unperturbed system.

We notice that the fundamental propriety of those Floquet states is the fact that they form a complet set of time-dependent solutions in the Hilbert space. So, the global state $|\Psi(t)\rangle$ of the quantum system can be written in the form of a linear combination of the $|\Psi_n(t)\rangle$:

$$|\Psi(t)\rangle = \sum_n a_n|\Psi_n(t)\rangle$$

where the a_n are time-independent coefficients.

2.2. *Jump operators*

We introduce the operators $A(t)$ and $A^+(t)$ which are responsible for the instantaneous transitions between the Floquet states. Their effects on the Floquet states are similar to those of the conventional annihilation and creation operators. These operators must satisfy the conditions[3]

$$A(t)|\psi_n(t)\rangle = \sqrt{n}|\psi_{n-1}(t)\rangle \tag{9}$$

$$A^+(t)|\psi_n(t)\rangle = \sqrt{n+1}|\psi_{n+1}(t)\rangle \qquad (10)$$

So, their expressions are defined by the relations

$$A(t) = e^{i\frac{\Delta\varepsilon}{\hbar}t}T(t)aT^+(t) \qquad (11)$$

$$A^+(t) = e^{-i\frac{\Delta\varepsilon}{\hbar}t}T(t)a^+T^+(t) \qquad (12)$$

where $\Delta\varepsilon$ is the difference between two successive Floquet levels:

$$\Delta\varepsilon = \varepsilon_{n+1} - \varepsilon_n \qquad (13)$$

The Floquet decomposition (Eq. (2)) gives us different possibilities to solve the evolution equation of the periodically-time-dependent harmonic oscillator. However, the determination of the corresponding Floquet operators is not unique, while the Floquet theory supposes the existence of an infinite number of couples $(R, T(t))$, solutions of equation (2), depending on the form of the perturbed Hamiltonian. To determine these two operators, we used the technique of the resonating averages method elaborated by G.Lochak,[2] based on the generalized Bogoliubov approximation.

2.3. *Resonating averages method (RAM)*

The principle of this method consists in the separation of the perturbed Hamiltonian $H_I(t)$, written in the interaction picture (Eq. (5)), into an averaging part, $\overline{H_I}(t)$, and an oscillating part $\widetilde{H_I}(t)$, such as :

$$H_I(t) = \overline{H_I}(t) + \frac{d\widetilde{H_I}(t)}{dt} \qquad (14)$$

with

$$\overline{H_I}(t) = \sum_{k=0}^{n-1}(H_I)_k e^{iw_k t} \qquad (15)$$

and

$$\widetilde{H_I}(t) = \sum_{k=n}^{\infty}(H_I)_k \frac{e^{iw_k t}}{iw_k} \qquad (16)$$

where $(H_I)_k$ are a set of constant hermitian operators, and w_k a frequency sequence from which one can extract a subsequence which contains the frequency $w_0 = 0$.

The RAM stipulates that if one supposes that this subsequence contains a set of a finite number n of frequencies as $\{w_0, w_1, ..., w_{n-1}\}$, the other frequencies $\{w_n, w_{n+1}, ...etc\}$ are not harmonic or harmonic combinations of the n first frequencies.

2.3.1. *First order and first order ameliorated solutions*

The application of the RAM to Eq. (4), to first order of μ, gives the differential equation

$$i\hbar\frac{d^{(1)}V_I(t)}{dt} = \mu\overline{H_I}(t)^{(1)}V_I(t) \tag{17}$$

With the initial condition ${}^{(1)}V_I(t_0) = 1$, one has

$${}^{(1)}U_I(t) =^{(1)} V_I(t) =^{(1)} T(t)e^{-i^{(1)}Rt/\hbar} \tag{18}$$

The first order ameliorated solution is defined as

$${}^{(1a)}U_I(t) = [1 - \frac{i\mu}{\hbar}\widetilde{H_I}(t)]^{(1)}V_I(t) \tag{19}$$

2.3.2. *Second order and second order ameliorated solutions*

The second order fundamental solution of Eq. (4) is defined by the operator

$${}^{(2)}U_I(t) = [1 - \frac{i\mu}{\hbar}\widetilde{H_I}(t)]\Gamma(t) \tag{20}$$

where $\Gamma(t)$ is an unitary operator solution of :

$$i\hbar\frac{d\Gamma(t)}{dt} = \{\mu\overline{H_I}(t) + \frac{i\mu^2}{\hbar}[\overline{S_I}(t) + \frac{\overline{Z_I}(t)}{2}]\}\Gamma(t) \tag{21}$$

where

$$S_T(t) = \widetilde{H_I}\overline{H_I} - \overline{H_I}\widetilde{H_I} \tag{22}$$

$$Z_I(t) = \widetilde{H_I}\frac{d\widetilde{H_I}}{dt} - \frac{d\widetilde{H_I}}{dt}\widetilde{H_I} \tag{23}$$

One may define the second order ameliorated solution as

$${}^{(2a)}U_I(t) = [1 - \frac{i\mu}{\hbar}\widetilde{H_I}(t) + \mu^2 A_2(t)]\Gamma(t) \tag{24}$$

where the operator $A_2(t)$ is given by

$$A_2(t) = \frac{1}{\hbar^2}\{\widetilde{S_I}(t) + \frac{\widetilde{Z_I}(t)}{2} - \frac{\widetilde{H_I^2}(t)}{2}\} \tag{25}$$

Determination of first and second order ameliorated solutions ${}^{(1a)}U_1(t)$ and ${}^{(2a)}U_1(t)$ (Eqs. (19) and (24)) enables to obtain the first and second order couples $({}^{(1a)}R,{}^{(1a)}T(t))$, $({}^{(2a)}R,{}^{(2a)}T(t))$ and Floquet-states $|{}^{(1a)}\psi_n(t)\rangle, |{}^{(2a)}\psi_n(t)\rangle$, respectively.

3. Applications

3.1. *Forced harmonic oscillator (FHO)*

We have applied the above method, and compared our results to those of other studies,[39]-[12] which have used other methods, to a FHO with the following Hamiltonian

$$H(t, \nu_0) = \frac{p^2}{2m_0} + \frac{1}{2} m_0 \omega_0^2 q^2 + \mu \sin(\nu_0 t) q \tag{26}$$

where m_0 and ω_0 are the constants describing the mass and the frequency of the unperturbed oscillator, respectively.
By introducing the annihilation operator a

$$a = \frac{1}{\sqrt{2\hbar}} (\sqrt{\beta_0} q + i \frac{p}{\sqrt{\beta_0}}) \tag{27}$$

where

$$\beta_0 = m_0 \omega_0 \tag{28}$$

the quantized form of $H(t)$ is

$$H(t, \nu_0) = \hbar\omega_0 (a^+ a + \frac{1}{2}) + \mu \sqrt{\frac{\hbar}{2\beta_0}} (a + a^+) \sin \nu_0 t = H_0 + \mu H_1(t) \tag{29}$$

We applied the RAM to the perturbed Hamiltonian $H_I(t)$, expressed in the interaction picture as Eq. (5), choosing a frequency sequence such as

$$\omega_k = 0,\ \nu_0 - 2\omega_0,\ -(\nu_0 - 2\omega_0) \tag{30}$$

we deduced from Eq. (29) the averaging and oscillating parts of $H_I(t)$, given respectively by

$$\overline{H_I}(t) = 0 \tag{31}$$

and

$$\frac{d\widetilde{H_I}(t)}{dt} = \sqrt{\frac{\hbar}{2\beta_0}} \sin \nu_0 t (e^{-i\omega_0 t} a + e^{i\omega_0 t} a^+) \tag{32}$$

Then

$$\widetilde{H_I}(t) = \alpha_0(t) a + \alpha_0^*(t) a^+ \tag{33}$$

where

$$\alpha_0(t) = \frac{-1}{2} \sqrt{\frac{\hbar}{2\beta_0}} (\frac{e^{i(\nu_0 - \omega_0)t}}{\nu_0 - \omega_0} + \frac{e^{-i(\nu_0 + \omega_0)t}}{\nu_0 + \omega_0}) \tag{34}$$

3.1.1. *First order calculations*

The fundamental first order ameliorated solution of Eq. (4) is obtained as

$$^{(1a)}U(t) = \{1 - \frac{iu}{\hbar}(\alpha_1(t)a + \alpha_1^*(t)a^+)\}e^{-i\frac{H_0 t}{\hbar}} \tag{35}$$

where

$$\alpha_1(t) = e^{i\omega_0 t}\alpha_0(t) \tag{36}$$

From Eq. (2), we deduced the Floquet operators such as

$$^{(1a)}T(t) = 1 - \frac{iu}{\hbar}(\alpha_1(t)a + \alpha_1^*(t)a^+) \tag{37}$$

$$^{(1a)}R = H_0 \tag{38}$$

The ameliorated first order operator $^{(1a)}R$ is simply equal to the unperturbed Hamiltonian H_0.
This absence of first order linear effects in is one of the characteristic properties of the Floquet Hamiltonian. This fact follows directly from the perturbative approach applied to the unitary operator (Eq. (19)). However, it was shown that the phenomenon is deeper for a large class of field pulses which are not necessarily monochromatic, and all odd perturbative corrections must vanish[13]-,[1426]
Then, the eigenstates of the Floquet operator $^{(1a)}R$ are obtained as

$$|^{(1a)}\Phi_n(t)\rangle = e^{-i\frac{^{(1a)}Rt}{\hbar}}|n\rangle = e^{-i(n+\frac{1}{2})\omega_0 t}|n\rangle \tag{39}$$

and the Floquet states are given by the following expression

$$|^{(1a)}\Psi_n(t)\rangle = e^{-i\frac{^{(1a)}\varepsilon_n t}{\hbar}}\{|n\rangle - \frac{i\mu}{\hbar}[\alpha_1(t)\sqrt{n}|n-1\rangle + \alpha_1^*(t)\sqrt{n+1}|n+1\rangle]\} \tag{40}$$

where

$$^{(1a)}\varepsilon_n = \hbar\omega_0(n + \frac{1}{2}) \tag{41}$$

To determine the expectation values and the uncertainty relation of the position and impulsion operators, one can write

$$q = \sqrt{\frac{\hbar}{2\beta_0}}(a + a^+) \tag{42}$$

$$p = \sqrt{\frac{\hbar\beta_0}{2}}(a^+ - a)$$

The first ameliorated order expectation values of q, p, q^2 and p^2 in the Floquet states, given by Eq. (40), are

$$\langle^{(1a)}q\rangle = \frac{-\mu \sin \nu_0 t}{m_0(\nu_0^2 - \omega_0^2)} \tag{43}$$

$$\langle^{(1a)}p\rangle = \frac{-\mu\nu_0 \cos \nu_0 t}{\nu_0^2 - \omega_0^2} \tag{44}$$

$$\langle^{(1a)}q^2\rangle = \frac{\hbar}{2\beta_0}\{2n+1+\frac{\mu^2}{\hbar^2}[2n(n+1)Re(\alpha_1^2)+((2n+1)^2+2)|\alpha_1|^2]\} \tag{45}$$

$$\langle^{(1a)}p^2\rangle = \frac{\hbar\beta_0}{2}\{2n+1+\frac{\mu^2}{\hbar^2}[-2n(n+1)Re(\alpha_1^2)+((2n+1)^2+2)|\alpha_1|^2]\} \tag{46}$$

The explicit expression of the uncertainty product is

$$^{(1a)}\langle\Delta q\Delta p\rangle = \frac{\hbar}{2}(2n+1)\{1+\mu^2\frac{(2n+1)}{\hbar\beta_0(\nu_0^2-\omega_0^2)^2}(\nu_0^2\cos^2\nu_0 t+\omega_0^2\sin^2\nu_0 t)\}^{1/2} \tag{47}$$

3.1.2. *Second order calculations*

To solve the differential equation (21), we begin by expressing the operator $Z(t)$ in the form

$$Z(t) = 2Im(\alpha_0(t)\frac{d\alpha_0^*(t)}{dt}) \tag{48}$$

Then, we deduce its averaging and oscillating parts as

$$\overline{Z}(t) = \frac{-i\hbar}{2m_0(\nu_0^2-\omega_0^2)} \tag{49}$$

$$\frac{d\widetilde{Z}(t)}{dt} = \frac{i\hbar(e^{2i\nu_0 t}+e^{-2i\nu_0 t})}{4m_0(\nu_0^2-\omega_0^2)} \tag{50}$$

Because $H_I = 0$ and $S_I = 0$,Eq. (21) is reduced to

$$\frac{i\hbar d\Gamma(t)}{dt} = \frac{i\mu^2}{2\hbar}\overline{Z}(t)\Gamma(t) \tag{51}$$

So, we deduce

$$\Gamma(t) = e^{\frac{-i\mu^2 t}{4\hbar m_0(\nu_0^2-\omega_0^2)}} \tag{52}$$

Then, the second order evolution operator is given by

$$^{(2)}U(t) = \{1-\frac{iu}{\hbar}(\alpha_1(t)a+\alpha_1^*(t)a^+)\}e^{-i\frac{H_0 t}{\hbar}}\Gamma(t) \tag{53}$$

and the corresponding Floquet components are

$$^{(2)}R = \hbar\omega_0(a^+a + \frac{1}{2}) + \frac{\mu^2}{4m_0(\nu_0^2 - \omega_0^2)} \tag{54}$$

$$^{(2)}T(t) = 1 - \frac{iu}{\hbar}(\alpha_1(t)a + \alpha_1^*(t)a^+) =^{(1a)} T(t) \tag{55}$$

The ameliorated second order solution is

$$^{(2a)}U_I(t) = \{1 - \frac{iu}{\hbar}\widetilde{H_I}(t) + \mu^2 A_2(t)\}\Gamma(t) \tag{56}$$

with

$$A_2(t) = \frac{1}{\hbar^2}(\widetilde{S}(t) + \frac{\widetilde{Z}(t) - \widetilde{H_I^2}(t)}{2}) \tag{57}$$

By integrating Eq. (50), we get

$$\widetilde{Z}(t) = \frac{\hbar(e^{2i\nu_0 t} - e^{2i\nu_0 t})}{8m_0\nu_0(\nu_0^2 - \omega_0^2)} \tag{58}$$

then,

$$\begin{aligned} ^{(2a)}U(t) = \{1 - \tfrac{iu}{\hbar}(\alpha_1(t)a + \alpha_1^*(t)a^+) - \tfrac{u^2}{2\hbar^2}(\varphi(t) \\ +\alpha_1^2(t)a^2 + \alpha_1^{*2}(t)a^{+2} + |\alpha_1|^2(2a^+a + 1))\}e^{-i^{(2a)}Rt/h} \end{aligned} \tag{59}$$

where

$$\varphi(t) = \frac{-i\hbar\sin\nu_0 t\cos\nu_0 t}{2m_0\nu_0(\nu_0^2 - \omega_0^2)} \tag{60}$$

The ameliorated second order Floquet-Hamiltonian has the following form

$$^{(2a)}R = \hbar\omega_0(a^+a + \frac{1}{2}) + \frac{\mu^2}{4m_0(\nu_0^2 - \omega_0^2)} \tag{61}$$

$$\begin{aligned} ^{(2a)}T = 1 - \tfrac{iu}{\hbar}(\alpha_1(t)a + \alpha_1^*(t)a^+) - \tfrac{u^2}{2\hbar^2}(\varphi(t) + \alpha_1^2(t)a^2 \\ +\alpha_1^{*2}(t)a^{+2} + |\alpha_1|^2(2a^+a + 1)) \end{aligned} \tag{62}$$

In the same way, we determined the ameliorated second order Floquet states

$$|^{(2a)}\Psi_n(t)\rangle =^{(2a)} T|^{(2a)}\Phi_n(t)\rangle \tag{63}$$

where

$$|^{(2a)}\Phi_n(t)\rangle = e^{-i^{(2a)}Rt/\hbar}|n\rangle = e^{-i^{(2a)}\varepsilon_n t/\hbar}|n\rangle \tag{64}$$

and the quasi-energy is

$$^{(2a)}\varepsilon_n = \hbar\omega_0(n + \frac{1}{2}) + \frac{\mu^2}{4m_0(\nu_0^2 - \omega_0^2)} \tag{65}$$

The explicit expression of the Floquet states is

$$|^{(2a)}\Psi_n(t)\rangle = \{[1+\varphi(t) - \frac{\mu^2}{2\hbar^2}(2n+1)|\alpha_1|^2]|n\rangle \\ -\frac{i\mu}{\hbar}[\alpha_1(t)\sqrt{n}|n-1\rangle + \alpha_1^*(t)\sqrt{n+1}|n+1\rangle] \tag{66}$$

$$-\frac{\mu^2}{2\hbar^2}[\alpha_1^2(t)\sqrt{n(n-1)}|n-2\rangle + \alpha_1^{*2}(t)\sqrt{(n+1)(n+2)}|n+2\rangle]\}e^{-i\frac{^{(2a)}\varepsilon_n t}{\hbar}}$$

Let us notice that the expression of the ameliorated second order Floquet operator $^{(2a)}T(t)$, given by Eq. (62), is comparable to the so-called coherent states generator published by Breuer et al.,[3] and by Fox et al.,[9] as well as the ameliorated second order Floquet states. The detailed comparisons with these works accompanied with some discussions are given in section **3.1.4**.

Moreover, the expression of the quasi-energy (Eq. (65)) is identical to that given in some other works,[39]-[13] which supports the observations previously made. Consequently, the corrections are absent to first order but appear, when the approximations are pushed to second order, and are expressed by a shift D_ε given by

$$D_\varepsilon = \frac{\mu^2}{4m_0(\nu_0^2 - \omega_0^2)} \tag{67}$$

which does not depend on the quantum number n, but on the strength μ and the pulsation ν_0 of the interacting field. This means effect of the perturbation is as a whole spectral shift of all Floquet levels.

This $\Delta\varepsilon$ is giving by

$$\Delta\varepsilon = \varepsilon_{n+1} - \varepsilon_n \tag{68}$$

The sense of displacement depends on the sign of D_ε.If $D_\varepsilon > 0$, the levels shift up and inversely they shift down if $D_\varepsilon < 0$.
Therefore, the frequency of a transition of the Floquet states, which corresponds to an absorption or emission of the so-called "Floquet quanta",[14] is equal to that of the stationary states. So, we can write

$$^{(2a)}\Delta\varepsilon = \hbar^{(2a)}\omega_F \tag{69}$$

and

$$^{(2a)}\omega_F = \omega_0 \tag{70}$$

The corresponding expectation values of the position and momentum operators are given by

$$\langle^{(2a)}q\rangle = \sqrt{\frac{\hbar}{2\beta_0}}\frac{i\mu}{\hbar}\{2Im(\alpha_1) + \frac{\mu^2}{\hbar^2}(2n+1)[-\varphi(t)Re(\alpha_1) + |\alpha_1|^2 Im(\alpha_1)]\} \tag{71}$$

$$\langle^{(2a)}p\rangle = -\sqrt{\frac{\hbar}{2\beta_0}}\frac{\mu}{\hbar}\{2Re(\alpha_1) + \frac{\mu^2}{\hbar^2}(2n+1)[-\varphi(t)Im(\alpha_1) + |\alpha_1|^2 Re(\alpha_1)]\} \tag{72}$$

$$\langle^{(2a)}q^2\rangle = \frac{\hbar}{2\beta_0}\{(2n+1) - \frac{4\mu^2}{\hbar^2}Im(\alpha_1) + \frac{\mu^4}{\hbar^4}(2n+1)[\frac{|\varphi(t)|^2}{4} + 2\varphi(t)Re(\alpha_1)Im(\alpha_1) \tag{73}$$

$$+\frac{5(2n^2+2n+3)}{4}Re^4(\alpha_1) + \frac{(2n^2+2n+7)}{4}Im^4(\alpha_1)$$

$$-\frac{(6n^2+6n+11)}{2}Re^2(\alpha_1)Im^2(\alpha_1)]\}$$

$$\langle^{(2a)}p^2\rangle = \frac{\hbar\beta_0}{2}\{(2n+1) + \frac{4\mu^2}{\hbar^2}Re(\alpha_1) + \frac{\mu^4}{\hbar^4}(2n+1)[\frac{|\varphi(t)|^2}{4} - 2\varphi(t)Re(\alpha_1)Im(\alpha_1) \tag{74}$$

$$+\frac{5(2n^2+2n+3)}{4}Im^4(\alpha_1) + \frac{(2n^2+2n+7)}{4}Re^4(\alpha_1)$$

$$-\frac{(6n^2+6n+11)}{2}Re^2(\alpha_1)Im^2(\alpha_1)]\}$$

and the explicit uncertainty product is

$$\langle^{(2a)}\Delta q\Delta p\rangle = \frac{\hbar}{2}(2n+1)\{1 + \frac{\mu^4}{8\hbar^2 m_0^2(\nu_0^2-\omega_0^2)^2}[\frac{\sin^2 2\nu_0 t}{2\nu_0^2} \tag{75}$$

$$+\frac{3(2n^2+2n+1)}{\omega_0^2(\nu_0^2-\omega_0^2)^2}(\omega_0^2\sin^2\nu_0 t + \nu_0^2\cos^2\nu_0 t)]\}$$

3.1.3. *Jump operators*

In section **2.2**, we have defined the operators $A(t)$ and $A^+(t)$ which are called jump operators,[315] From Eqs. (11) and (12), we can write, to first order in μ

$$A(t) = e^{i\frac{^{(1a)}\Delta\varepsilon}{\hbar}t}\ {}^{(1a)}T(t)a^{(1a)}T^+(t) \tag{76}$$

$$A^+(t) = e^{-i\frac{^{(1a)}\Delta\varepsilon}{\hbar}t} \ ^{(1a)}T(t)a^{+(1a)}T^+(t) \tag{77}$$

By substituting the expression (37) of the unitary operator $^{(1a)}T(t)$, we obtain the two conjugal operators such as,

$$A(\alpha_1, t) = a + \frac{i\mu}{\hbar}\alpha_1^*(t) \tag{78}$$

$$A^+(\alpha_1, t) = a^+ - \frac{i\mu}{\hbar}\alpha_1(t) \tag{79}$$

The action of those ladder operators on the Floquet states $|^(1a)\psi_n(t)\rangle$ is equivalent to the action of the annihilation and creation operators on the stationary states. These operators are time dependent in the Schrödinger picture. Their expressions (78) and (79) are exactly identical to those published by Breuer et al.[3]

The operators $A(t)$ and $A^+(t)$ satisfy the commutation relation $[A(t), A^+(t)] = 1$
One can also give an hermitical symmetric operator[15] $N(t)$ such as

$$N(t) = \frac{1}{2}(AA^+ + A^+A) \tag{80}$$

From the definitions

$$A(t)|\psi_n(t)\rangle = \sqrt{n}|\psi_{n-1}(t)\rangle \tag{81}$$

$$A^+(t)|\psi_n(t)\rangle = \sqrt{n+1}|\psi_{n+1}(t)\rangle \tag{82}$$

we have

$$A^+A|\psi_n(t)\rangle = n|\psi_n(t)\rangle \tag{83}$$

and

$$AA^+|\psi_n(t)\rangle = (n+1)|\psi_n(t)\rangle \tag{84}$$

Then, the exact action of the operator $N(t)$ on the Floquet states is defined as

$$N(t)|\psi_n(t)\rangle = (n + \frac{1}{2})|\psi_n(t)\rangle \tag{85}$$

This operator $N(t)$ is equivalent to the number operator of the unperturbed harmonic oscillator.

3.1.4. *Comparisons and discussions*

We have presented in the above paragraphs some analytical results of Floquet operators, Floquet states, expectations values and uncertainties products of the position and momentum operators. To test the validity of our approach and to confirm the accuracy of our calculations, we will give some direct comparisons with other works given in the literature,[39]-[12] . In particular, Breuer et al.,[3] have generalized the stochastic wave-function method to the description of open dissipative systems in strong Laser field, and have employed the Floquet representation for quantum systems with time-periodic Hamiltonian, among which the forced harmonic oscillator, submitted to a Laser field with an Hamiltonian of the same form of Eq. (26).

Fox et al.[9] have presented a non perturbative method based on a quasi-adiabatic method. This approach has been applied to an harmonic oscillator in an electric field of the form,

$$E_F(t) = -eE_0\sqrt{\frac{\hbar}{2m_0\omega_0}}\cos\omega t(a + a^+) \tag{86}$$

To compare our results to those of these authors, we should make some transformations by using the trigonometric formulation $\cos x = \sin(x + \frac{\Pi}{2})$.Then, in all our trigonometric functions (sinus or cosines) we add a phase $(+\frac{\Pi}{2})$ i.e. (our results $\longrightarrow^{phase+\frac{\Pi}{2}}$ Fox results)

i)<u>Annihilation and position operators in the Heisenberg picture:</u>

Breuer et al.[3] have calculated the expression of annihilation and position operators, for the FHO in the Heisenberg picture, to first order in μ. Therefore, we shall use the evolution operator obtained in Eq. (35) to express $a_H(t)$ and $q_H(t)$.So, we can write

$$a_H(t) =^{(1a)} U^+(t)a^{(1a)}U(t) \tag{87}$$

and

$$q_H(t) =^{(1a)} U^+(t)q^{(1a)}U(t) \tag{88}$$

we obtain

$$a_H(t) = ae^{-i\omega_0 t} + \frac{i\mu'}{2}(\frac{e^{i\nu_0 t}}{\nu_0 + \omega_0} + \frac{e^{-i\nu_0 t}}{\nu_0 - \omega_0}) \tag{89}$$

and

$$q_H(t) = \sqrt{\frac{\hbar}{2\beta_0}}\{ae^{-i\omega_0 t} + a^+ e^{i\omega_0 t} + i\mu^{'}\omega_0 \frac{e^{i\nu_0 t} - e^{-i\nu_0 t}}{\omega_0^2 - \nu_0^2}\} \tag{90}$$

with

$$\mu^{'} = \frac{\mu}{\sqrt{2\hbar\beta_0}} \tag{91}$$

These expressions are identical to those, $a_H(t,t_0)$ and $x_H(t,t_0)$, published by Breuer et al.[3] using the initial condition $t_0 = 0$ in the integration of the operator $\frac{d\widetilde{H}_I(t)}{dt}$ in Eq. (32) and putting $\widetilde{H}(0) = 0$ and $\hbar = 1$ With those conditions, we can obtain exact expressions of $a_H(t,t_0)$ and $x_H(t,t_0)$ given in reference.[3] But in our approach, we need to have general forms without imposing any initial condition on our quantum system.

ii)<u>Floquet operator and Floquet states</u>

Let us notice that our ameliorated first order operator ${}^{(1a)}T(t)$ and ameliorated second order one ${}^{(2a)}T(t)$ correspond to a first and a second order development of the Breuer's operator $e^{\varphi(t)}D(w)$ and the Fox's operator $e^{\varphi_F(t)}D_F(\eta)$, respectively (but for this latter we must take the trigonometric transformation previously indicated).

Where the so-called coherent state generator[3] is defined by

$$D(w) = exp(wa^+ - w^*a) \tag{92}$$

By comparing the Breuer's variable $w(t)$ with our variable $\alpha_1(t)$ of Eq. (36), and the Fox's one $\eta(t)$, we have the equality

$$w(t) = \frac{-i\mu}{\hbar}\alpha_1^*(t)(Eq.(116) of [3]) \overset{\nu_0 t + \frac{\pi}{2}}{\longrightarrow} the\ Fox's\ variable\ \eta(t)\ (Eq.(35) of [9]) \tag{93}$$

as well as the phase $\varphi(t)$ obtained in Eq. (60) is directly comparable to the Breuer's one $\varphi_B(t)$ (see the Eq. (120) of reference[3]) and exactly the same with the Fox's one $\varphi_F(t)$ (see Eq. (34) of reference[9]).

iii)<u>Quasi-energy</u>

The Floquet energy given by Eq. (65) is identical to that calculated by Breuer et al.,[3] Fox et al.,[9] Lefebvre et al.,[10] Maitra et al.[11] and Samal et al.[12]

iv) Expectation values and uncertainty products

The expectation values of the position q and momentum p operators are exactly comparable to those given to first order in μ by Fox et al.,[9] but to second order, our values contain a second order correction term in μ . Finally, our uncertainty product contains a 2^{nd} order correction term in μ for the case of the ameliorated first order uncertainty relation, while in the case of ameliorated second order one, it contains a 4^{th}order correction term in μ.

v) Jump operators

Let us notice that the expressions of jump operators given by Eqs. (78) and (79) are identical to those published by Breuer et al.[3]

3.2. *Pulsating Harmonic Oscillator*

An other good candidate for the application of this formalism is the pulsating Harmonic Oscillator described by the periodically time-dependent Hamiltonian, which is a particular form of the Caldirola-Kanai Hamiltonian[16]-[19]

$$H(q,p,t) = \frac{p^2}{2m(t)} + \frac{1}{2}m(t)\omega_0^2 q^2 \tag{94}$$

where ω_0 is the constant oscillator frequency and $m(t)$ is a periodically time-varying mass such as[18]

$$m(t) = m_0 e^{2\mu \sin \nu t} \tag{95}$$

where μ is the strength of the pulsation and is the mass frequency. We shall find it convenient to introduce the canonical transformations

$$Q = q\sqrt{\frac{m}{m_0}} \qquad P = p\sqrt{\frac{m_0}{m}} \tag{96}$$

The Hamiltonian becomes

$$H(Q,P,t) = \frac{p^2}{2m_0} + \frac{1}{2}m_0\omega_0^2 Q^2 + \frac{\dot{m}}{m}\frac{PQ+QP}{2} \tag{97}$$

From the form of Eq. (1), one can deduce the quantized forms of the unperturbed and perturbed Hamiltonians, respectively, such as

$$H_0 = \hbar\omega_0(a^+a + \frac{1}{2}) \tag{98}$$

$$H_1(t) = -i\hbar\nu \cos \nu t(a^2 - a^{+2}) \tag{99}$$

where the annihilation operator a is defined as

$$a = \frac{1}{\sqrt{2\hbar\beta_0}}(\beta_0 Q + iP) \tag{100}$$

By using Eqs. (5) and (14) and Eqs. (15) and (16) and choosing a frequency sequence such as $\omega_k = 0, \nu - 3\omega_0, -(\nu - 3\omega_0)$
we deduce

$$\overline{H_I} = 0 \tag{101}$$

$$\frac{d\widetilde{H_I}(t)}{dt} = -i\hbar\nu \cos \nu t(e^{-2i\omega_0 t}a^2 - e^{2i\omega_0 t}a^{+2}) \tag{102}$$

Integration of Eq. (102) gives

$$\widetilde{H_I}(t) = \gamma_0(t)a^2 + \gamma_0^*(t)a^{+2} \tag{103}$$

where

$$\gamma_0(t) = -\frac{\hbar\nu}{\nu^2 - 4\omega_0^2}e^{-2i\omega_0 t}(2\omega_0 \cos \nu t + i\nu \sin \nu t) \tag{104}$$

3.2.1. *First order solutions*

i)Floquet operators

By solving Eq. (4), using Eq. (103) and from Eq. (19), one can write

$${}^{(1a)}U_1(t) = \left(1 - \frac{i\mu}{\hbar}e^{-i\omega_0 t a^+ a}\widetilde{H}_I e^{i\omega_0 t a^+ a}\right)e^{-iH_0 t/\hbar} ={}^{(1a)} T(t)e^{-i^{(1a)}Rt/\hbar} \tag{105}$$

and deduce the first order ameliorated Floquet components such as

$${}^{(1a)}R = H_0 \tag{106}$$

$${}^{(1a)}T(t) = 1 - \frac{i\mu}{\hbar}\left(\gamma_1(t)a^2 + \gamma_1^*(t)a^{+2}\right) \tag{107}$$

with

$$\gamma_1(t) = e^{2i\omega_0 t}\gamma_0(t) \tag{108}$$

For a comparison with other works, we will calculate the position operator in the Heisenberg picture, by using the first order ameliorated evolution operator of Eq. (105). So one may write

$$Q_H(t) ={}^{(1a)} U_1^+(t)Q^{(1a)}U_1(t) \tag{109}$$

To first order of μ we get

$$Q_H(t) = Q(0)\cos\omega_0 t + \frac{P(0)}{\beta_0}\sin\omega_0 t + \frac{4\mu\nu}{4\omega_0^2 - \nu^2}\times \tag{110}$$

$$\left\{Q(0)\left[\omega_0\left(2\cos^2\frac{\nu t}{2} - 1\right)\sin\omega_0 t - \nu\sin\frac{\nu t}{2}\cos\frac{\nu t}{2}\cos\omega_0 t\right]\right.$$

$$\left.+\frac{P(0)}{\beta_0}\left[\omega_0\left(2\sin^2\frac{\nu t}{2} - 1\right)\cos\omega_0 t - \nu\sin\frac{\nu t}{2}\cos\frac{\nu t}{2}\sin\omega_0 t\right]\right\}$$

$$with\ Q(0) = Q \quad and \quad P(0) = P. \tag{111}$$

If we take into account the initial condition $\widetilde{H}(0) = 0$ in the integration of Eq. (102), this result is rather comparable with that obtained by Abdalla et al.[18] which have employed the method of the perturbation theory. The comparison is also verified for $P_H(t)$ and $a_H(t)$.

ii)Floquet states and uncertainty relation

From Eqs. (6) and (8) and using Eqs. (106) and (107), we have calculated the first order Floquet states,

$$|^{(1a)}\psi_n(t)\rangle = e^{-i(n+1/2)\omega_0 t}\left\{|n\rangle - \frac{i\mu}{\hbar}\left[\sqrt{n(n-1)}\gamma_1(t)|n-2\rangle\right.\right.$$
$$\left.\left.+\sqrt{(n+1)(n+2)}\gamma_1^*(t)|n+2\rangle\right]\right\} \tag{112}$$

The expectation values of q, p, q^2 and p^2 are

$$\langle q\rangle = 0 \tag{113}$$

$$\langle p\rangle = 0 \tag{114}$$

$$\langle^{(1a)}q^2\rangle = \frac{\hbar}{2\beta}(2n+1)\{1 + \mu F_1(n,\nu,t) + \mu^2 F_2(n,\nu,t)\} \tag{115}$$

$$\langle^{(1a)}p^2\rangle = \frac{\hbar\beta}{2}(2n+1)\{1 - \mu F_1(n,\nu,t) + \mu^2 F_2(n,\nu,t)\} \tag{116}$$

where

$$\beta(t) = m(t)\omega_0 \tag{117}$$

$$F_1(n,\nu,t) = \lambda_0 \sin\nu t \tag{118}$$

$$F_2(n,\nu,t) = (n^2+n+5)\frac{\lambda_1}{2}(\lambda_0^2\sin^2\nu t + \cos^2\nu t) \tag{119}$$

with

$$\lambda_0 = \frac{\nu}{2\omega_0} \tag{120}$$

and

$$\lambda_1 = \frac{4\nu_0^2}{\omega_0^2 - 1} \tag{121}$$

Finally, calculation of the uncertainty relation in the Floquet-states ${}^{(1a)}|\psi_n(t)\rangle$ gives

$$^{(1a)}(\Delta q\Delta p) = \frac{\hbar}{2}(2n+1)\Big(1+\mu^2\ {}_{(1a)}F(n,\nu,t)\Big)^{\frac{1}{2}} \tag{122}$$

$$_{(1a)}F(n,\nu,t) = \frac{\lambda_1^2}{4\lambda_0^2}\Big\{\Big(n^2+n+1\Big)\Big(\lambda_0^2\sin^2\nu t+\cos^2\nu t\Big)+4\cos^2\nu t\Big\} \tag{123}$$

3.2.2. *Second order solutions*

i)Floquet operators

Solving the differential equation (21) and using Eqs. (101) and (103) we deduce

$$\Gamma(t) = e^{-i\mu^2\lambda_1\frac{H_0 t}{\hbar}} \tag{124}$$

Then, the second order solution (Eq. (20)) is obtained in the following form

$$^{(2)}U(t) = \Big(1-\frac{i\mu}{\hbar}e^{-i\omega_0 ta^+a}\widetilde{H}_I e^{i\omega_0 ta^+a}\Big)e^{-i(1+\mu^2\lambda_1)H_0 t/\hbar} = ^{(2)}T(t)e^{-i\ {}^{(2)}Rt/\hbar} \tag{125}$$

Consequently, the second order Floquet operators are

$$^{(2)}R = \Big(1+\mu^2\lambda_1\Big)H_0 \tag{126}$$

$$^{(2)}T(t) = 1-\frac{i\mu}{\hbar}\Big(\gamma_1(t)a^2+\gamma_1^*(t)a^{+2}\Big) \tag{127}$$

One observes that a μ-second order correction appears in ${}^{(1a)}R$. Therefore, the ameliorated second order evolution operator is

$$ {}^{(2a)}U(t) = \left(1 + e^{-iH_0t/\hbar}\left\{\left(-\frac{i\mu}{\hbar}\right)\widetilde{H}_I + \mu^2 A_2(t)\right\}e^{iH_0t/\hbar}\right)e^{-i(1+\mu^2\lambda_1)H_0t/\hbar} \tag{128} $$

because

$$ \widetilde{S}_I(t) = 0 \tag{129} $$

then

$$ A_2(t) = -\frac{1}{2\hbar^2}\left\{f_0(t)H_0 + \left(\gamma_0(t)a^2 + \gamma_0^*(t)a^{+2}\right)^2\right\} \tag{130} $$

where

$$ f_0(t) = i\hbar\lambda_1\frac{\sin 2\nu t}{\nu} \tag{131} $$

Then, we deduce

$$ {}^{(2a)}R = {}^{(2)}R \tag{132} $$

$$ {}^{(2a)}T(t) = 1 - \frac{i\mu}{\hbar}\left(\gamma_1(t)a^2 + \gamma_1^*(t)a^{+2}\right) - \frac{\mu^2}{2\hbar^2}\left\{f_0(t)H_0 + 2|\gamma_1(t)|^2\left[\frac{H_0^2}{\hbar_0^2\omega_0^2} + \frac{3}{4}\right]\right. \tag{133} $$

$$ \left. + \gamma_1^2(t)a^4 + \gamma_1^{*2}(t)a^{+4}\right\} $$

A second order correction term in μ is also present in the operator ${}^{(2a)}T(t)$

ii)<u>Floquet states</u>

The eigenstates of the operator ${}^{(2a)}R$ have the following form

$$ {}^{(2a)}|\phi_n\rangle = e^{-i(n+\frac{1}{2})(1+\mu^2\lambda_1^2)\omega_0 t}|n\rangle \tag{134} $$

From Eq. (6) and using Eq. (133), the construction of the ameliorated second order Floquet states leads to

$$ \begin{aligned} {}^{(2a)}|\psi_n\rangle = e^{-i(n+\frac{1}{2})(1+\mu^2\lambda_1^2)\omega_0 t}\Big\{ & C_0^n(t)|n\rangle + C_{-2}^n(t)|n-2\rangle \\ & + C_2^n(t)|n+2\rangle + C_{-4}^n(t)|n-4\rangle + C_4^n(t)|n+4\rangle\Big\} \end{aligned} \tag{135} $$

where the n-time-functions are

$$ C_0^n(t) = 1 - \frac{\mu^2}{2\hbar^2}\left[\left(n + \frac{1}{2}\right)\hbar\omega_0 f_0(t) + 2(n^2 + n + 1)|\gamma_1(t)|^2\right] \tag{136} $$

$$C^n_{-2}(t) = -\frac{i\mu}{\hbar}\sqrt{n(n-1)}\gamma_1(t) \tag{137}$$

$$C^n_2(t) = -\frac{i\mu}{\hbar}\sqrt{(n+1)(n+2)}\gamma_1^*(t) \tag{138}$$

$$C^n_{-4}(t) = -\frac{\mu^2}{2\hbar^2}\sqrt{n(n-1)(n-2)(n-3)}\gamma_1^2(t) \tag{139}$$

$$C^n_4(t) = -\frac{\mu^2}{2\hbar^2}\sqrt{(n+1)(n+2)(n+3)(n+4)}\gamma_1^{*2}(t) \tag{140}$$

iii)Heisenberg uncertainty relation

The expectation values of q , p and q^2 in the $|^{(2a)}\psi_n(t)\rangle$ states are

$$^{(2a)}\langle q\rangle =^{(2a)} \langle p\rangle = 0 \tag{141}$$

$$\langle^{(2a)}q^2\rangle = \frac{\hbar}{2\beta}(2n+1)\Big\{1+\mu M_1(\nu,t)+\mu^2 M_2(\nu,t)+\mu^3 M_3(\nu,t)+\mu^4 M_4(\nu,t)\Big\} \tag{142}$$

$$\langle^{(2a)}p^2\rangle = \frac{\hbar\beta}{2}(2n+1)\Big\{1-\mu M_1(\nu,t)+\mu^2 M_2(\nu,t)-\mu^3 M_3(\nu,t)+\mu^4 M_4(\nu,t)\Big\} \tag{143}$$

where

$$M_1(\nu,t) = \frac{4i}{\hbar}Im(\gamma_1(t)) \tag{144}$$

$$M_2(\nu,t) = 2(n^2+n+4)\frac{|\gamma_1(t)|^2}{\hbar^2} \tag{145}$$

$$M_3(\nu,t) = \frac{i}{\hbar^3}\Big\{4(n^2+n+5)|\gamma_1(t)|^2 Im\gamma_1 - (n^2+n+1)\hbar\omega_0 Re(\gamma_1)f_0(t)\Big\} \tag{146}$$

$$M_4(\nu,t) = \frac{N_m(n)}{\hbar^4}|\gamma_1^2(t)|^2 - (2n+1)^2\left(\frac{\omega_0 f_0(t)}{4\hbar}\right)^2 \tag{147}$$

with

$$N_m(n) = \frac{1}{4}(n^2+n+2)(n^2+n+52) + (n^2+n+1)^2 + 1 \tag{148}$$

The second order uncertainty product in Floquet-states ${}^{(2a)}|\psi_n(t)\rangle$ is obtained in the following form

$$^{(2a)}(\Delta q\Delta p) = \frac{\hbar}{2}(2n+1)\Big(1+\mu^2\chi_2(n,\nu,t)+\mu^4\chi_4(n,\nu,t)\Big)^{\frac{1}{2}} \tag{149}$$

where

$$\chi_2(n,\nu,t) = \frac{4}{\hbar^2}\left[(n^2+n+4)Re^2\Big(\gamma_1(t)\Big) - n(n+1)Im^2\Big(\gamma_1(t)\Big)\right] \tag{150}$$

$$\chi_4(n,\nu,t) = (2n+1)^2\frac{(\omega_0|f_0(t)|)^2}{8\hbar^2} + \frac{2}{\hbar^4}\left[N_m(n)+2(n^2+n+4)^2\Big(|\gamma_1(t)|^2\Big)^2\right] \tag{151}$$

3.2.3. *Jump operators*

From Eqs. (11) and (12) we have determined the ladder operators in the following forms

$$A(t) = e^{i\frac{\Delta\varepsilon}{\hbar}t}\left(a+\frac{2i\mu}{\hbar}\gamma_1^*(t)a^+\right) \tag{152}$$

$$A^+(t) = e^{-i\frac{\Delta\varepsilon}{\hbar}t}\left(a^+-\frac{2i\mu}{\hbar}\gamma_1(t)a\right) \tag{153}$$

where $\gamma_1(t)$ is given by Eq. (108)

3.3. *Harmonic oscillator with a periodical frequency*

We also apply our approach to an Harmonic Oscillator with a periodically time-varying frequency $\Omega(t)$,[5],[14][27]-.[30] We consider the Hamiltonian of the form

$$H(t) = \frac{p^2}{2m_0} + \frac{1}{2}m_0\Omega^2(t)q^2 \tag{154}$$

By choosing a Mathieu frequency,[5],[14],[25][27]-[30] or the so-called experiment's Paul trap frequency,[14],[23],[27][29]-[31] such as

$$\Omega^2(t) = \omega_0^2(1+\mu\cos 2\omega t) \tag{155}$$

where μ and ω are the amplitude and the frequency of the oscillations, respectively. Then

$$H(t) = \frac{p^2}{2m_0} + \frac{1}{2}m_0\omega_0^2q^2 + \frac{\mu}{2}m_0\omega_0^2\cos 2\omega t q^2 \tag{156}$$

The quantized form of the perturbation is

$$H_1(t) = \frac{\hbar\omega_0}{4}\cos 2\omega t(a^2 + a^{+2} + 2aa^+ + 1) \tag{157}$$

The RAM applied to the interaction picture form of $H_1(t)$ (Eq. (5)) gives

$$\overline{H}_I = 0 \tag{158}$$

$$\widetilde{H}_I = \eta_0(t)a^2 + \eta_0^*(t)a^{+2} + \hbar\eta_2(t)\left(aa^+ + \frac{1}{2}\right) \tag{159}$$

where

$$\eta_0(t) = \frac{-i\hbar\omega_0}{8}e^{-2i\omega_0 t}\left(\frac{e^{2i\omega t}}{\omega - \omega_0} - \frac{e^{-2i\omega t}}{\omega + \omega_0}\right) \tag{160}$$

$$\eta_2(t) = \frac{\omega_0 \sin 2\omega t}{4\omega} \tag{161}$$

Let us remark that our approach is very simple and the Hamiltonian operator of this case is comparable to the one published by Profilo et al.[5] In fact, if we write $\widetilde{H}_I(t)$ in the standard picture we have,

$$\widetilde{H}_1(t) = \eta_1(t)a^2 + \eta_1^*(t)a^{+2} + \hbar\eta_2(t)\left(aa^+ + \frac{1}{2}\right) \tag{162}$$

with

$$\eta_1(t) = \eta_1(t)e^{2i\omega_0 t} \tag{163}$$

By differencing this operator, we have

$$\frac{d\widetilde{H}_1(t)}{dt} = \dot{\eta}_1(t)a^2 + \dot{\eta}_1^*(t)a^{+2} + \hbar\dot{\eta}_2(t)\left(aa^+ + \frac{1}{2}\right) \tag{164}$$

where

$$\dot{\eta}_1(t) = \frac{\hbar\omega_0\omega}{2\left(\omega^2 - \omega_0^2\right)}\Big[\omega\cos(2\omega t) + 2i\sin(2\omega t)\Big] \tag{165}$$

By replacing it in the total Hamiltonian, we obtain

$$\widehat{H}(t) = \hbar\Omega'(t)\left(a^+a + \frac{1}{2}\right) + \frac{\mu\hbar\omega_0\omega}{2\left(\omega^2 - \omega_0^2\right)}\Big\{\Big[\omega\cos(2\omega t) + 2i\omega_0\sin(2\omega t)\Big]a^2 + \Big[\omega\cos(2\omega t) - 2i\omega_0\sin(2\omega t)\Big]a^{+2}\Big\} \tag{166}$$

where

$$\Omega'(t) = \omega_0\left(1 + \frac{\mu}{2}\cos 2\omega t\right) \tag{167}$$

which is a first order development of $\Omega(t)$ given in Eq. (155). We see that this Hamiltonian, where 2ω represents the driven frequency, resembles that appearing in the context of electromagnetic and acoustic parametric interactions, and is similar to that obtained by Profilo and al.[5] using the invariant method in the framework of the Lie Group.

3.3.1. *First order Floquet operators and Floquet states*

From Eq. (19) and with the initial condition ${}^{(1)}V_I(t_0 = 1$, one has

$$^{(1a)}U_1(t) = \left(1 - \frac{i\mu}{\hbar}e^{-iH_0t/\hbar}\widetilde{H}_I e^{iH_0t/\hbar}\right)e^{-iH_0t/\hbar} \tag{168}$$

From the decomposition of Eq. (2), we determined the first order ameliorated Floquet components such as

$$^{(1a)}R = H_0 \tag{169}$$

$$^{(1a)}T(t) = 1 - \frac{i\mu}{\hbar}\widetilde{H}_1(t) \tag{170}$$

where $\widetilde{H}_1(t)$ is given by Eq. (162).
We also remark that, to first order approximation, the operator ${}^{(1a)}R$ is equal to the simple Oscillator Hamiltonian, and that the operator ${}^{(1a)}T(t)$ is unitary and periodic with the same period as $H_1(t)$.
From Eqs. (6) and (8) and using Eqs. (162) and (170), the first order Floquet states are obtained as

$$|^{(1a)}\psi_n(t)\rangle = e^{-i(n+1/2)\omega_0 t}\left\{\left[1 - i\mu\eta_2(t)\left(n + \frac{1}{2}\right)\right]|n\rangle \right. \tag{171}$$

$$\left. -\frac{i\mu}{\hbar}\left[\sqrt{n(n-1)}\eta_1(t)|n-2\rangle + \sqrt{(n+1)(n+2)}\eta_1^*(t)|n+2\rangle\right]\right\}$$

Then, the wave-Floquet function may be written in the form

$$^{(1a)}\psi_n(q,t) = e^{-i(n+1/2)\omega_0 t}\left\{\left[1 - i\mu\eta_2(t)\left(n + \frac{1}{2}\right)\right]\varphi_n(q) \right. \tag{172}$$

$$\left. -\frac{i\mu}{\hbar}\left[\sqrt{n(n-1)}\eta_1(t)\varphi_{n-2}(q) + \sqrt{(n+1)(n+2)}\eta_1^*(t)\varphi_{n+2}(q)\right]\right\}$$

where

$$\varphi_n(q) = \left(\frac{\beta_0}{\hbar\pi}\right)^{\frac{1}{2}} \frac{1}{\sqrt{2^n n!}} e^{-\beta_0 q^2/2\hbar} H_n\left(\sqrt{\frac{\beta_0}{\hbar}} q\right) \tag{173}$$

is the wave function of the free oscillator, and H_n is the Hermite polynomials.

The expectation values of q, p, q^2 and p^2 are deduced, and one has

$$\langle q\rangle_{n,n} = \langle p\rangle_{n,n} = 0 \tag{174}$$

$$\langle^{(1a)} q^2\rangle_{n,n} = \frac{\hbar}{2\beta_0}(2n+1)\Big\{1 + \mu\xi_1(t) + \mu^2\xi_2(n,t)\Big\} \tag{175}$$

$$\langle^{(1a)} p^2\rangle_{n,n} = \frac{\hbar\beta_0}{2}(2n+1)\Big\{1 - \mu\xi_1(t) + \mu^2\xi_2(n,t)\Big\} \tag{176}$$

where

$$\xi_1(t) = \frac{\omega_0^2}{2\left(\omega^2 - \omega_0^2\right)^2} \cos(2\omega t) \tag{177}$$

$$\xi_2(n,t) = \frac{\omega_0^2}{32\left(\omega^2 - \omega_0^2\right)} \left\{5\left(n^2+n+\frac{3}{2}\right)\sin^2(2\omega t) - 2\frac{\omega_0^2}{\omega^2}\left(n^2+n+\frac{1}{4}\right)\sin^2(2\omega t)\right. \tag{178}$$

$$\left. +(n^2+n+5)\frac{\omega_0^2}{\omega^2 - \omega_0^2}\right\}$$

$$\xi_2'(n,t) = \frac{\omega_0^2}{32\left(\omega^2 - \omega_0^2\right)} \left\{\left(n^2+n+\frac{7}{2}\right)\sin^2(2\omega t) - 2\frac{\omega_0^2}{\omega^2}\left(n^2+n+\frac{1}{4}\right)\sin^2(2\omega t)\right. \tag{179}$$

$$\left. +(n^2+n+5)\frac{\omega_0^2}{\omega^2 - \omega_0^2}\right\}$$

The first order uncertainty relation in Floquet-states $^{(1a)}|\psi_n(t)\rangle$ is obtained in the form

$$^{(1a)}\Big(\Delta q\Delta p\Big)_{n,n} = \frac{\hbar}{2}(2n+1)\Big(1 + \mu^2 F_n(t)\Big)^{\frac{1}{2}} \tag{180}$$

$$F_n(t) = \frac{\omega_0^2}{16\left(\omega^2 - \omega_0^2\right)} \left\{\left(3n^2 + 3n + \frac{11}{4} - 2\left(n + \frac{1}{2}\right)^2 \frac{\omega_0^2}{\omega^2}\right)\sin^2(2\omega t)\right. \tag{181}$$

$$+\frac{\omega_0^2}{(\omega^2-\omega_0^2)}\Big(n^2+n+1+4\sin^2(2\omega t)\Big)\Big\}$$

The Heisenberg uncertainty condition $\Delta q\Delta p \geq \frac{\hbar}{2}$ is satisfied for all time t.

3.3.2. *Second order Floquet operators and Floquet states*

Solving the differential equation (21) and using Eqs. (22), (23, (158) and (159), we deduce

$$\Gamma(t) = e^{-i\mu^2\frac{\omega_0^2 H_0 t}{16\hbar(\omega^2-\omega_0^2)}} \tag{182}$$

Then, the second order evolution operator Eq. (20) is obtained in the form

$$^{(2)}U(t) = \Big(1-\frac{i\mu}{\hbar}\widetilde{H}_1(t)\Big)e^{-i\Big(1+\mu^2\frac{\omega_0^2}{16(\omega^2-\omega_0^2)}\Big)H_0 t/\hbar} \tag{183}$$

The corresponding Floquet operators are

$$^{(2)}R = \Big(1+\mu^2\frac{\omega_0^2}{16(\omega^2-\omega_0^2)}\Big)H_0 \tag{184}$$

$$^{(2)}T(t) =^{(1a)} T(t) \tag{185}$$

The determination of the ameliorated second order solution (Eq. (24)) necessitates the calculation of the operators $\widetilde{Z}_I(t)$ and $\widetilde{H}_I^2(t)$. We get

$$\widetilde{Z}_I(t) = 2\hbar^2\Big[\delta_0(t)\Big(2a^+a+1\Big)+\delta_2'(t)a^2-\delta_2'^*(t)a^{+2}\Big] \tag{186}$$

with

$$\delta_0(t) = \frac{-i\omega_0^3}{2^7\omega\Big(\omega^2-\omega_0^2\Big)}\sin(4\omega t) \tag{187}$$

$$\delta_2'(t) = \frac{\omega_0^2}{2^7}\left[\frac{\omega_0 e^{-4i\omega t}}{\omega(\omega+\omega_0)(2\omega+\omega_0)}-\frac{\omega_0 e^{4i\omega t}}{\omega(\omega-\omega_0)(2\omega-\omega_0)}+\frac{2}{\omega^2-\omega_0^2}\right]e^{-2i\omega_0 t} \tag{188}$$

and

$$\widetilde{H}_I^2(t) = \eta_0^2(t)a^4+\eta_0^{*2}(t)a^{+4}+\Big(\hbar^2\eta_2^2(t)+2|\eta_0^2(t)|^2\Big)\Big(a^+a+\frac{1}{2}\Big)^2+\frac{3}{2}|\eta_0^2(t)|^2 \tag{189}$$

$$+\hbar\eta_0(t)\eta_2(t)\Big(2a^+a+3\Big)a^2+\hbar\Big(\eta_0(t)\eta_2(t)\Big)^*a^{+2}\Big(2a^+a+3\Big)$$

The operator $A_2(t)$, given by Eq. (25), has the following form

$$A_2(t) = \frac{1}{2\hbar^2}\Big(\widetilde{Z}_I(t) - \widetilde{H}_I^2(t)\Big) \tag{190}$$

Therefore, the ameliorated second order evolution operator has the form

$$^{(2a)}U(t) = \left\{1 + e^{-iH_0t/\hbar}\left(-\frac{i\mu}{\hbar}\widetilde{H}_I + \mu^2 A_2(t)\right)e^{iH_0t/\hbar}\right\}e^{-i\left(1+\frac{\mu^2\omega_0^2}{16(\omega^2-\omega_0^2)}\right)H_0t/\hbar} \tag{191}$$

One may deduce the Floquet operators such as

$$^{(2a)}R =^{(2)} R \tag{192}$$

$$^{(2a)}T(t) =^{(2)} T(t) + \mu^2 A_2'(t) \tag{193}$$

with

$$A_2'(t) = e^{-iH_0t/\hbar}A_2(t)e^{iH_0t/\hbar} \tag{194}$$

Here the second order correction of has affected the operator $^{(2)}T(t)$. According to Eq. (6), the second order Floquet states are given by

$$|^{(2a)}\psi_n(t)\rangle =^{(2a)} T(t)|^{(2a)}\phi_n(t)\rangle \tag{195}$$

where $|^{(2a)}\phi_n(t)\rangle$, the eigenstates of operator $^{(2a)}R$, are such as

$$|^{(2a)}\phi_n(t)\rangle = e^{-i(n+\frac{1}{2})(1+\frac{\mu^2\omega_0^2}{16(\omega^2-\omega_0^2)})\omega_0 t}|n\rangle \tag{196}$$

The construction of the ameliorated second order Floquet states gives

$$|^{(2a)}\psi_n(t)\rangle = e^{-i\frac{^{(2a)}\varepsilon_n(\omega)t}{\hbar}}\Big\{C_0^n(t)|n\rangle + C_{-2}^n(t)|n-2\rangle + C_2^n(t)|n+2\rangle$$
$$+C_{-4}^n(t)|n-4\rangle + C_4^n(t)|n+4\rangle\Big\} \tag{197}$$

where the second order quasi-energy is given by

$$^{(2a)}\varepsilon_n(\omega) = \hbar\omega_0\left(n+\frac{1}{2}\right)\left(1+\frac{\mu^2\omega_0^2}{16(\omega^2-\omega_0^2)}\right) \tag{198}$$

The n-time-dependent coefficients are such as

$$C_0^n(t) = 1 - i\mu\left(n+\frac{1}{2}\right)\eta_2(t) + \mu^2\left[-\frac{3}{2\hbar^2}|\eta_1(t)|^2 + (2n+1)\delta_0(t) + (2n+1)^2\delta_1(t)\right] \tag{199}$$

$$C_{-2}^n(t) = \sqrt{n(n-1)}\left\{\frac{-i\mu}{\hbar}\eta_1(t) + \mu^2\Big[\delta_2(t) + (2n-1)\delta_3(t)\Big]\right\} \tag{200}$$

$$C_2^n(t) = \sqrt{(n+1)(n+2)}\left\{\frac{-i\mu}{\hbar}\eta_1^*(t) + \mu^2\Big[-\delta_2^*(t) + (2n+3)\delta_3^*(t)\Big]\right\} \quad (201)$$

$$C_{-4}^n(t) = \mu^2\sqrt{n(n-1)(n-2)(n-3)}\delta_4(t) \quad (202)$$

$$C_4^n(t) = \mu^2\sqrt{(n+1)(n+2)(n+3)(n+4)}\delta_4^*(t) \quad (203)$$

with

$$\delta_1(t) = \frac{-1}{4}\left(\frac{2}{\hbar^2}|\eta_1(t)|^2 + \eta_2^2(t)\right) \quad (204)$$

$$\delta_2(t) = \delta_2'(t)e^{2i\omega_0 t} \quad (205)$$

$$\delta_3(t) = -\frac{1}{2\hbar}\eta_1(t)\eta_2(t) \quad (206)$$

$$\delta_4(t) = \frac{-\eta_1^2(t)}{2\hbar^2} \quad (207)$$

The expectation values of q, p, q^2 and p^2 in the $|^{(2a)}\psi_n(t)\rangle$ states are such as

$$^{(2a)}\langle q\rangle_{n,n} =^{(2a)} \langle p\rangle_{n,n} = 0 \quad (208)$$

$$\langle^{(2a)}q^2\rangle_{n,n} = \frac{\hbar}{2\beta_0}(2n+1)\left\{1 + \mu f_1^n(t) + \mu^2 f_{2+}^n(t) + \mu^3 f_{3+}^n(t) + \mu^4 f_{4+}^n(t)\right\} \quad (209)$$

$$\langle^{(2a)}p^2\rangle_{n,n} = \frac{\hbar\beta_0}{2}(2n+1)\left\{1 - \mu f_1^n(t) + \mu^2 f_{2-}^n(t) + \mu^3 f_{3-}^n(t) + \mu^4 f_{4-}^n(t)\right\} \quad (210)$$

where

$$f_1^n = \frac{4Re(i\eta_1(t))}{\hbar} \quad (211)$$

$$f_{2\pm}^n = (2n^2+2n+7)\frac{|\eta_1|^2}{\hbar^2} + \left(n+\frac{1}{2}\right)^2\eta_2^2 + 2(2n+1)^2\delta_1 \mp 4Re(\delta_2) \quad (212)$$

$$\pm 4(n^2+n+3)Re(\delta_3) \mp \frac{2i}{\hbar}(n^2+n+1)\eta_2 Im(i\eta_1)$$

$$f_{3\pm}^n(t) = (2n+1)^2\eta_2\Big(i\delta_0 \mp 2Re(i\delta_3)\Big) + \frac{4i}{\hbar}(n^2+n+5)Im(\eta_1^*\delta_2) \quad (213)$$

$$\pm\frac{16}{\hbar}(n^2+n+3)Re(i\eta_1\delta_4^*)+\frac{20}{\hbar}(2n^2+2n+3)Re(i\eta_1\delta_3^*)$$

$$\mp\frac{4}{\hbar}\left[(2n+1)^2\delta_1-\frac{3|\eta_1|^2}{2\hbar^2}\right]Re(i\eta_1)$$

$$\pm 4(n^2+n+1)\left(\delta_0\frac{Im(i\eta_1)}{\hbar}+i\eta_2\left(\frac{Im(\delta_2)}{2}-Im(\delta_3)\right)\right)$$

$$f_{4\pm}^n(t)=\frac{9}{4\hbar^4}\left(|\eta_1|^2\right)^2\mp\frac{6|\eta_1|^2}{\hbar^2}Re\Big((n^2+n+3)\delta_3-\delta_2\Big)+2(n^2+n+5)|\delta_2|^2+ \tag{214}$$

$$N_0(n)|\delta_3|^2+(2n+1)^2\bigg(-\delta_0^2\pm 4Re(\delta_0\delta_3)-\frac{3|\eta_1|^2}{\hbar^2}\delta_1\pm 4\delta_1 Re\Big((n^2+n+3)\delta_3-\delta_2\Big)$$

$$+(2n+1)^2\delta_1^2\bigg)-20(2n^2+2n+3)Re(\delta_2\delta_3^*)\pm(4n^2+n+1)\delta_0 Im(2\delta_3-\delta_2)$$

$$\mp 16(n^2+n+3)Re(\delta_2\delta_4^*)+N_1(n,6)|\delta_4|^2\pm 2N_1(n,4)Re(\delta_3\delta_4^*)$$

where

$$N_0(n)=32n(n+1)+2(4n^2+4n+5)(n^2+n+9) \tag{215}$$

$$N_1(n,k)\frac{(n+4)!}{n!}+\frac{n!}{(n-4)!}+2^k(n^2+n+3) \tag{216}$$

Then, the second order uncertainty relation in Floquet-states ${}^{(2a)}|\psi_n(t)\rangle$ has the form

$${}^{(2a)}(\Delta q\Delta p)_{n,n}=\frac{\hbar}{2}(2n+1)\Big(1+\mu^2\chi_2^n(\nu,t)+\mu^3\chi_3^n(\nu,t)+\mu^4\chi_4^n(\nu,t)\Big)^{\frac{1}{2}} \tag{217}$$

where

$$\chi_2^n(\nu,t)=(2n+1)^2\left[4\delta_1(t)+\frac{\eta_2^2(t)}{2}\right]+\frac{2}{\hbar^2}(2n^2+2n+7)|\eta_1(t)|^2-\frac{16}{\hbar^2}(Re(i\eta_1(t))^2 \tag{218}$$

$$\chi_3^n(\nu,t)=2i(2n+1)^2\eta_2(t)\delta_0(t)+\frac{8i}{\hbar}(n^2+n+5)Im(\eta_1^*(t)\delta_2(t))+ \tag{219}$$

$$\frac{40}{\hbar}(2n^2+2n+3)Re(i\eta_1(t)\delta_3^*(t))+\frac{8}{\hbar}Re(i\eta_1(t))\Big\{4Re(\delta_2(t))$$

$$-4(n^2+n+3)Re(\delta_3(t))+\frac{2i}{\hbar}(n^2+n+1)\eta_2(t)Im(i\eta_1(t))\Big\}$$

$$\chi_4^n(\nu,t) = (2n+1)^2\Big\{\eta_2^2\Big[\frac{3}{16}(2n+1)^2\eta_2^2 + \Big(3n^2+3n+\frac{1}{2}\Big)\frac{|\eta_1^2|}{\hbar^2} + \tag{220}$$

$$\frac{8i}{\hbar}Im(\eta_1)Re\Big(n^2+n+3\Big)\delta_3 - \delta_2\Big)\Big] - 2\delta_0^2 - \frac{32i}{\hbar}Im(\eta_1)Re(\delta_0)Re(\delta_3)\Big\}$$

$$+4(n^2+n+1)\Big\{(n^2+n+1)\Big[\frac{2}{\hbar^4}(|\eta_1|^2)^2 - \frac{\eta_2^2 Re^2(\eta_1)}{\hbar^2}\Big] + \Big[\frac{16i}{\hbar^3}Im(\eta_1)|\eta_1|^2$$

$$-\frac{4}{\hbar}Re(\eta_1)\eta_2\Big]Re((n^2+n+3)\delta_3-\delta_2)+Re^2(\delta_2)+\frac{8i}{\hbar}Im(\eta_1)\Big[Re(\delta_0)Im(\delta_2-2\delta_3)$$

$$+Im(\delta_0)Im(\delta_2)\Big]\Big\} + \frac{4}{\hbar^4}(n^2+n-3)^2(|\eta_1|^2)^2 - 4(n^2+n+5)Im^2(\delta_2)$$

$$+\Big[2N_0(n) - 16(n^2+n+3)^2\Big]Re^2(\delta_3) - 2N_0 Im^2(\delta_3) + 2N_1(n,6)|\delta_4|^2$$

$$-40(2n^2+2n+3)Im(\delta_3)Im(\delta_2) + \frac{16iIm(\eta_1)}{\hbar}\Big\{8(n^2+n+3)\Big[Re(\delta_2)Re(\delta_4)$$

$$+Im(\delta_2)Im(\delta_4)\Big]\Big\} - N_1(n,4)\Big[Re(\delta_3)Re(\delta_4) - Im(\delta_3)Im(\delta_4)\Big]\Big\}$$

$$-24(2n^2+2n+1)\Big[Re(\delta_3)Re(\delta_2) + \frac{4i}{\hbar}Im(\eta_1)Im(\delta_0)Im(\delta_3)\Big]$$

3.4. *Harmonic Oscillator with time-dependent mass and frequency*

The last system considered in this study is the time-dependent Harmonic Oscillator described by the periodically time-varying Hamiltonian such as

$$H(t) = \frac{p^2}{2m(t)} + \frac{1}{2}m(t)\Omega^2(t)q^2 \tag{221}$$

where $\Omega(t)$ and $m(t)$ are the time-varying frequency and mass respectively, given by

$$\Omega^2(t) = \omega^2(t)(1+\mu f(t)) \tag{222}$$

$$m(t) = m_0 e^{\mu\eta(t)} \tag{223}$$

where μ is the strength of variation of $\Omega(t)$ and $m(t)$
The equations $\eta(t)$ and $f(t)$ are two periodically time-varying functions satisfying

$$\eta(t+T) = \eta(t) \tag{224}$$

$$f(t+T) = f(t)$$

we introduce the canonical transformations

$$Q = e^{\frac{\mu\eta(t)}{2}} q \qquad P = e^{\frac{-\mu\eta(t)}{2}} p \tag{225}$$

The Hamiltonian becomes

$$H(Q,P,T) = \frac{P^2}{2m_0} + \frac{1}{2} m_0 \omega_0^2 Q^2 + \mu\dot{\eta}(t)\frac{PQ+QP}{2} + \frac{\mu}{2} m_0 \omega_0^2 f(t) Q^2 \tag{226}$$

we introduce the annihilation and creation operators a and a^+ such as

$$a = \frac{1}{\sqrt{2\hbar\beta_0}}(\beta_0 Q + iP) \tag{227}$$

where

$$\beta_0 = m_0\omega_0 \tag{228}$$

By writing the Hamiltonian (Eq. (226)) in the quantized form of Eq. (1), one can deduce the unperturbed and perturbed Hamiltonians respectively as

$$H_0 = \hbar\omega_0(a^+a + \frac{1}{2}) \tag{229}$$

$$H_1(t) = \frac{\hbar\omega_0}{4} f(t)(a^2 + a^{+2} + 2aa^+ + 1) - \frac{i\hbar}{2}\dot{\eta}(t)(a^2 - a^{+2}) \tag{230}$$

choosing the functions

$$\eta(t) = 2\sin\nu t \tag{231}$$

$$f(t) = \cos 2\omega t$$

In the interaction picture form of $H_1(t)$, and using the RAM technique, one may deduce the averaging and oscillating parts respectively as

$$\overline{H}_I = 0 \tag{232}$$

$$\widetilde{H}_I = \rho_0(t)a^2 + \rho_0^*(t)a^{+2} + \hbar\eta_2(t)\left(aa^+ + \frac{1}{2}\right) \tag{233}$$

where

$$\rho_0(t) = \left\{ \frac{-\hbar\nu}{2}\left(\frac{e^{i\nu t}}{\nu - 2\omega_0} - \frac{e^{i\nu t}}{\nu + 2\omega_0} \right) - \frac{i\hbar\omega_0}{16}\left(\frac{e^{2i\omega t}}{\omega - \omega_0} - \frac{e^{-2i\omega t}}{\omega + \omega_0} \right) \right\} e^{-2i\omega_0 t} \tag{234}$$

$$\eta_2(t) = \frac{\omega_0 \sin 2\omega t}{4\omega} \tag{235}$$

3.4.1. *First order solutions*

From Eq. (19), with the initial condition ${}^{(1)}V_I(t_0 = 1$, we have

$${}^{(1a)}U(t) = \left(1 - \frac{i\mu}{\hbar} e^{-i\omega_0 t a^+ a} \widetilde{H}_I e^{-i\omega_0 t a^+ a} \right) e^{iH_0 t/\hbar} =^{(1a)} T(t) e^{-i\ {}^{(1a)}Rt/\hbar} \tag{236}$$

The first order ameliorated Floquet operators are given by

$${}^{(1a)}R = H_0 \tag{237}$$

$${}^{(1a)}T(t) = 1 - i\mu\left(\rho_1(t) a^2 + \rho_1^*(t) a^{+2} + \eta_2(t)\left(aa^+ + \frac{1}{2} \right) \right) \tag{238}$$

where

$$\rho_1(t) = e^{2i\omega_0 t} \frac{\rho_0(t)}{\hbar} \tag{239}$$

We notice that, to first order approximation, the operator ${}^{(1a)}R$ is equal to the unperturbed Hamiltonian, and the ${}^{(1a)}T(t)$ operator is unitary and periodic with the same period of $H_1(t)$. From Eqs. (6) and (8) and using Eqs. (237) and (238), the first order Floquet states are such as

$$|{}^{(1a)}\psi_n(t)\rangle = e^{-i(n+1/2)\omega_0 t} \left\{ \left[1 - i\mu\eta_2(t)\left(n + \frac{1}{2} \right) \right\} |n\rangle - i\mu\left[\sqrt{n(n-)}\rho_1(t)|n-2\rangle \right. \right. \tag{240}$$

$$\left. \left. + \sqrt{(n+1)(n+2)}\rho_1^*(t)|n+2\rangle \right] \right\}$$

The expectation values of Q, P, Q^2 and P^2 are deduced and one gets

$$\langle Q\rangle_{n,n} = \langle P\rangle_{n,n} = 0 \tag{241}$$

$$\langle {}^{(1a)}Q^2\rangle_{n,n} = \frac{\hbar}{2\beta_0}(2n+1)\left\{ 1 + \mu\xi_1(t) + \mu^2\xi_{2+}(t) \right\} \tag{242}$$

$$\langle {}^{(1a)}P^2\rangle_{n,n} = \frac{\hbar\beta_0}{2}(2n+1)\left\{ 1 - \mu\xi_1(t) + \mu^2\xi_{2-}(t) \right\} \tag{243}$$

where

$$\xi_1(t) = 4iIm\Big(\rho_1(t)\Big) \tag{244}$$

$$\xi_{2\pm}(t) = \left(n+\frac{1}{2}\right)^2 \eta_2^2(t) + 2(n^2+n+5)|\rho_1(t)|^2 \pm 2(n^2+n+1)\eta_2(t)Re(\rho_1(t)) \tag{245}$$

The first order uncertainty relation in Floquet-states ${}^{(1a)}|\psi_n(t)\rangle$ is obtained in the following form

$${}^{(1a)}(\Delta q \Delta p)_{n,n} = \frac{\hbar}{2}(2n+1)\Big(1+\mu^2 F_{1a}(n,t)\Big)^{\frac{1}{2}} \tag{246}$$

$$F_{1a}(n,t) = 2\left(n+\frac{1}{2}\right)^2 \eta_2^2(t) + 4(n^2+n+5)Re^2(\rho_1(t)) - 4(n^2+n+1)Im^2(\rho_1(t)) \tag{247}$$

The Heisenberg uncertainty condition $\Delta q \Delta p \geq \frac{\hbar}{2}$ is satisfied for all time t.

3.4.2. *Second order solutions*

As $\overline{H}_I(t) = 0$ and $\overline{S}_I(t) = 0$, the differential equation (21) may be written in the following form

$$i\hbar\frac{d\Gamma}{dt} = \frac{i\mu^2}{2\hbar}\overline{Z}_I(t)\Gamma(t) \tag{248}$$

where

$$\Gamma(t) = e^{-i\mu^2\lambda_{m\Omega}\frac{H_0 t}{\hbar}} \tag{249}$$

and

$$\lambda_{m\Omega} = \frac{4\nu^2}{\nu^2 - 4\omega_0^2} + \frac{\omega_0^2}{16(\omega^2-\omega_0^2)} \tag{250}$$

Then, the second order evolution operator (Eq. (20)) is obtained in the following form

$${}^{(2)}U(t) = \left(1 - \frac{i\mu}{\hbar}e^{-i\frac{H_0 t}{\hbar}}\widetilde{H}_I e^{i\frac{H_0 t}{\hbar}}\right)e^{-i(1+\mu^2\lambda_{m\Omega})\frac{H_0 t}{\hbar}} \tag{251}$$

The corresponding Floquet operators are

$${}^{(2)}R = (1+\mu^2\lambda_{m\Omega})H_0 \tag{252}$$

$${}^{(2)}T(t) = {}^{(1a)}T(t) \tag{253}$$

Determination of the ameliorated second order solution (Eq. (24)) necessitates the calculation of the operators $\widetilde{Z}_I(t)$, $\widetilde{H}_I^2(t)$ such as

$$\widetilde{Z}_I(t) = \hbar^2\left[\varphi_0(t)\Big(2a^+a+1\Big) + \varphi_2'(t)a^2 - \varphi_2'^*(t)a^{+2}\right] \tag{254}$$

with

$$\varphi_0(t) = -i\hbar^2\omega_0\left\{\frac{2\nu\sin 2\nu t}{\nu^2-4\omega_0^2} + \frac{\omega_0^2\sin 4\omega t}{2^6\omega(\omega^2-\omega_0^2)} + \frac{\nu}{4(\nu^2-4\omega_0^2)(\omega^2-\omega_0^2)}\times\right. \tag{255}$$

$$\left.\left[\frac{(\nu-2\omega)(\nu\omega+2\omega_0^2)\cos(\nu+2\omega)t}{\nu+2\omega} - \frac{(\nu+2\omega)(\nu\omega-2\omega_0^2)\cos(\nu-2\omega)t}{\nu-2\omega}\right]\right\}$$

$$\varphi_2'(t) = \frac{\hbar^2\omega_0}{8\omega}e^{-2i\omega_0 t}\left\{\frac{\omega_0}{8}\left[\frac{\omega_0 e^{-4i\nu t}}{(\omega+\omega_0)(2\omega+\omega_0)} - \frac{\omega_0 e^{4i\nu t}}{(\omega-\omega_0)(2\omega-\omega_0)} + \frac{2\omega}{\omega^2-\omega_0^2}\right]\right. \tag{256}$$

$$+i\nu\left[\frac{(\nu+2(\omega-\omega_0))e^{i(\nu-2\omega)t}}{(\nu-2\omega_9)(\nu-2(\omega+\omega_0))} - \frac{(\nu-2(\omega+\omega_0))e^{i(\nu+2\omega)t}}{(\nu-2\omega_9)(\nu+2(\omega-\omega_0))}\right.$$

$$\left.\left.+\frac{(\nu+2(\omega+\omega_0))e^{-i(\nu-2\omega)t}}{(\nu+2\omega_9)(\nu-2(\omega-\omega_0))} - \frac{(\nu-2(\omega-\omega_0))e^{-i(\nu+2\omega)t}}{(\nu+2\omega_9)(\nu+2(\omega+\omega_0))}\right]\right\}$$

and

$$\widetilde{H}_I^2 = \rho_0^2(t)a^4 + \rho_0^{*2}(t)a^{+4} + |\rho_0(t)|^2(a^2a^{+2}+a^{+2}a^2) \tag{257}$$

The operator $A_2(t)$ given by Eq. (25) becomes

$$A_2(t) = \frac{1}{2\hbar^2}\Big(\widetilde{Z}_I(t) - \widetilde{H}_I^2(t)\Big) \tag{258}$$

So, the ameliorated second order evolution operator is given by

$${}^{(2a)}U(t) = \left\{1 + e^{-i\frac{H_0 t}{\hbar}}\left(-\frac{i\mu}{\hbar}\widetilde{H}_I + \mu^2 A_2(t)\right)e^{i\frac{H_0 t}{\hbar}}\right\}e^{-i(1+\mu^2\lambda_{m\Omega})\frac{H_0 t}{\hbar}} \tag{259}$$

One may deduce the following Floquet operators

$${}^{(2a)}R = {}^{(2)}R \tag{260}$$

$${}^{(2a)}T(t) = {}^{(2)}T(t) + \mu^2 A_2'(t) \tag{261}$$

with

$$A_2'(t) = e^{-i\frac{H_0 t}{\hbar}}A_2(t)e^{i\frac{H_0 t}{\hbar}} \tag{262}$$

then

$$^{(2a)}T(t) = 1 - i\mu\left[\rho_1 a^2 + \rho_1^* a^2 + \eta_2\left(a^+ a + \frac{1}{2}\right)\right] + \frac{\mu^2}{2}\times \tag{263}$$

$$\left\{\frac{2\varphi_0}{\hbar^2}\left(a^+a+\frac{1}{2}\right)-2|\rho_1|^2\left[\left(a^+a+\frac{1}{2}\right)^2+\frac{3}{4}\right]+\varphi_2 a^2-\varphi_2^* a^{+2}-\rho_1^2 a^4-\rho_1^{*2}a^{+4}\right\}$$

where

$$\varphi_2(t) = e^{2i\omega_0 t}\frac{\varphi_2'(t)}{\hbar^2} \tag{264}$$

According to Eq. (6), the second order Floquet states are

$$|^{(2a)}\psi_n(t)\rangle =^{(2a)} T(t)|^{(2a)}\phi_n(t)\rangle \tag{265}$$

where $|^{(2a)}\phi_n(t)\rangle$, the eigenstates of operator $^{(2a)}R$, have the following form

$$|^{(2a)}\phi_n(t)\rangle = e^{-i(n+\frac{1}{2})(1+\mu^2\lambda_{m\Omega})\omega_0 t}|n\rangle \tag{266}$$

By substituting Eq. (263) into Eq. (265), the construction of the ameliorated second order Floquet states gives

$$\begin{aligned} &|^{(2a)}\psi_n(t)\rangle = e^{-i\frac{\lambda_{m\Omega}t}{\hbar}}\{D_0^n(t)|n\rangle + D_{-2}^n(t)|n-2\rangle + D_2^n(t)|n+2\rangle \\ &+D_{-4}^n(t)|n-4\rangle + D_4^n(t)|n+4\rangle\} \end{aligned} \tag{267}$$

$$D_0^n(t) = 1-i\mu\left(n+\frac{1}{2}\right)\eta_2(t)+\mu^2\Big\{-(n^2+n+1)|\rho_1(t)|^2+(2n+1)\frac{\varphi_0 t}{2\hbar^2} \tag{268}$$

$$D_{-2}^n(t) = \sqrt{n(n-1)}\Big\{-i\mu\rho_1(t) + \mu^2\frac{\varphi_2(t)}{2}\Big\} \tag{269}$$

$$D_2^n(t) = \sqrt{(n+1)(n+2)}\Big\{-i\mu\rho_1^*(t) - \mu^2\frac{\varphi_2^*(t)}{2}\Big\} \tag{270}$$

$$D_{-4}^n(t) = \frac{-\mu}{2}\sqrt{n(n-1)(n-2)(n-3)}\rho_1^2(t) \tag{271}$$

$$D_4^n(t) = \frac{-\mu}{2}\sqrt{(n+1)(n+2)(n+3)(n+4)}\rho_1^{*2}(t) \tag{272}$$

The expectation values of q , p, q^2 and p^2 in the $|^{(2a)}\psi_n(t)$ states are such as

$$^{(2a)}\langle q\rangle_{n,n} =^{(2a)} \langle p\rangle_{n,n} = 0 \tag{273}$$

$$\langle^{(2a)} q^2\rangle_{n,n} = \frac{\hbar}{2\beta}(2n+1)\{1+\mu g_1^n(t)+\mu^2 g_{2+}^n(t)+\mu^3 g_{3+}^n(t)+\mu^4 g_{4+}^n(t)\} \quad (274)$$

$$\langle^{(2a)} p^2\rangle_{n,n} = \frac{\hbar\beta}{2}(2n+1)\{1-\mu g_1^n(t)+\mu^2 g_{2-}^n(t)+\mu^3 g_{3-}^n(t)+\mu^4 g_{4-}^n(t)\} \quad (275)$$

where

$$\beta(t) = \omega_0 m(t) \quad (276)$$

$$g_1^n(t) = 4iIm(\rho_1(t)) \quad (277)$$

$$g_{2\pm}^n(t) = 8|\rho_1|^2 + \Big(n+\frac{1}{2}\Big)\eta_2^2 \mp 2Re(\varphi_2) \pm 2(n^2+n+1)\eta_2 Re(\rho_1) \quad (278)$$

$$g_{3\pm}^n(t) = i(2n+1)^2\frac{\eta_2\varphi_0}{2\hbar} + 2i(n^2+n+5)\Big[Im(\rho_1^*\varphi_2) \pm 2Im(\eta_1)|\eta_1|^2\Big] \quad (279)$$

$$\pm 2i(n^2+n+1)\Big(\frac{\varphi_0 Re(\rho_1)}{\hbar 2} + \frac{\eta_2 Im(\varphi_2)}{2}\Big)$$

$$g_{4\pm}^n(t) = \left(n+\frac{1}{2}\right)^2 \frac{|\varphi_0|^2}{\hbar^2}\left[\frac{N_1(n,6)}{4} + (n^2+n+1)^2\right]|\rho_1|^4 + \frac{(n^2+n+5)}{2}|\varphi_2|^2 \quad (280)$$

$$\mp(n^2+n+1)\frac{\varphi_0 Im(\varphi_2)}{\hbar^2} \pm 2(3n^2+3n+7)Re(\varphi_2)Re^2(\rho_1)$$

$$\pm 2(n^2+n+5)Re(\varphi_2)Im^2(\rho_1) \mp 8(n^2+n+3)Im(\varphi_2)Re(\rho_1)Im(\rho_1)$$

and

$$N_1(n,k) = \frac{(n+4)!}{n!} + \frac{n!}{(n-4)!} + 2^k(n^2+n+3) \quad (281)$$

Then, the second order uncertainty relation in Floquet-states $^{(2a)}|\psi_n(t)\rangle$ has the following form

$$^{(2a)}(\Delta q\Delta p)_{n,n} = \frac{\hbar}{2}(2n+1)\Big(1+\mu^2 J_2^n(\nu,t)+\mu^3 J_3^n(\nu,t)+\mu^4 J_4^n(\nu,t)\Big)^{\frac{1}{2}} \quad (282)$$

where

$$J_2^n(\nu,t) = (2n+1)^2\frac{\eta_2^2}{2} + 16Re^2(\rho_1(t)) \quad (283)$$

$$J_3^n(\nu,t) = \frac{i}{\hbar}(2n+1)^2\eta_2\varphi_0 + 4i(n^2+n+5)Re(\rho_1)Im(\varphi_2) \quad (284)$$

$$-4i(n^2+n+1)Im(\rho_1)[Re(\varphi_2)+4\eta_2 Re(\rho_1)]$$

$$J_4^n(\nu,t) = (2n+1)^2\left[\frac{|\varphi_0|^2}{2\hbar^2}+4|\rho_1|^2\eta_2^2+\left(n+\frac{1}{2}\right)^2\frac{\eta_2^4}{4}\right]+(n^2+n+5)|\varphi_2|^2 \tag{285}$$

$$+\left(\frac{N_1(n,6)}{2}+2(n^2+n+1)^2+64\right)Re^4(\rho_1)-4Re^2(\varphi_2)$$

$$+\left(\frac{N_1(n,6)}{2}+2(n^2+n+1)^2-32(n^2+n+3)\right)Im^4(\rho_1)$$

$$-(N_1(n,6)-4(n^2+n+1)(n^2+n-7))Re^2(\rho_1)Im^2(\rho_1)$$

$$+8(n^2+n+1)\left[\frac{2\varphi_0 Re(\rho_1)Im(\rho_1)}{\hbar^2}+\eta_2 Im(\rho_1)Im(\varphi_2)\right]$$

$$-4(n^2+n+1)[(n^2+n+1)\eta_2^2 Re^2(\rho_1)+2\eta_2 Re(\rho_1)Re(\varphi_2)]$$

4. Conclusion

When the time variation of the hamiltonian of a quantum system is periodic, the Floquet theory is one of the most useful approaches which permits resolution of the Schrödinger equation. Therefore, in this paper, we have treated the Floquet decomposition, solution of the evolution equation, by using the resonating averages method from the first order in μ to the second ameliorated order. The approach was applied to the forced harmonic oscillator, to the harmonic oscillator with a time-dependent mass and a constant frequency, to the harmonic oscillator with a constant mass and a time-dependent frequency and to the harmonic oscillator with time-dependent mass and frequency. We have determined the approached forms of the evolution operator up to second ameliorated order in μ . Then, we have calculated the Floquet operators, Floquet states and verified that the uncertainty principle of Heisenberg is satisfied. Comparisons of our results, corresponding to each case of the above mentioned quantum systems, showed that they are identical to the results published by other authors which have used different methods,[3],[59]-,[1218] This perfect concordance showed the efficiency of our approach and the validity of the method.

References

1. J.H. Shirley , Phys. Rev. B 138 (1965) 979
2. G. Lochak etMumm Thioun , le journal de physique 30 (1969) 482
3. H.P. Breuer and F. Petruccione, Phys. Rev. A 55 (1997) 3101
4. H.R. Lewis Jr., Phys.Rev.Lett. 27 (1967) 510; J.Math. Phys. 9 (1968) 1976
5. G. Profilo and G. Soliana ,Phys. Rev. A 44 (1991) 2057
6. C. F. Lo, Am. J. Phys. 59(1991) 254-258 ; IL Nuovo Cimento 105 (1990) 497-506
7. K.H. Yeon, K.K. Lee, Ch.I. Um, T.F. George and L.N. Pandey,Phys.Rev. A48 (1993) 2716
8. S. Pepore, P. Winotaia, T.Osotchanb and U. Robkob, Sc. Asia 32 (2006) 173-179
9. R.F. Fox and L.V. Vela-Arevalo, Phys. Rev. A 66 (2002) 053402
10. R. Lefebvre and A. Palma, J. Mol. Struct. (Theochem)390 (1997) 23-32
11. Neepa T. Maitra and KieronBurke,Chem.Phys. Lett. 359 (2002) 237-240
12. Prasanjit Samaland Manoj K. Harbola, Chem. Phys. Lett. 433 (2006) 204-210
13. Sara Cruz y Cruz and Bogdan Mielnik, Phys. Lett. A352(2006) 36-40
14. Francisco Delgado C. and Bogdan Mielnik,Phys. Lett. A249 (1998) 369-375
15. M.L. Glasser, L.M.Nieto and B.F. Samsonov J. Phys. A: Math. Gen. 36 (2003) L1-L7
16. E. Kanai, Prog. Theor. Phys. 3 (1948) 440
17. P.Caldirola,Nuovo cimento 18 (1941) 393
18. M.S. Abdalla, R.K.Colegrave and A. Khorsravi , IL Nuovo Cimento 93 B (1986)
19. A. Geralico, G. Landolfi, G. Ruggeri and G. Soliani,arXiv:gr-qc/0308039 v2 31 Dec 2003
20. N . A. Lemos and C.P. Natividade, Nuovo Cimento 399 (1989) 211
21. S. Mandal,Opt. Comm. 240 (2004) 363-378
22. S. Mandal, Phys. Lett. A321 (2004) 308-318
23. W. Paul, Rev. Mod. Phys. 62 (1990)1947
24. Y. Achkar, S. Sayouri and A.L. Marrakchi, Phys.Chem. News 39 (2008) 19-24
25. A.Fedoul, A.L.Marrakchi,S.Sayouri and A.Chatwiti, Phys.Chem.News 24 (2005) 30-34
26. H. P. Breuer, K. Dietz and M. Holthaus, J. Phys. B: At.Mol. Opt. Phys. 24 (1991) 1343-1357
27. K.H. Yeon C. I. Um,and T. F. George, Phys. Rev.A 68 (2003) 052108-1-9
28. St.Weigert, J. Phys. A: Math. Gen 35 (2002) 4169
29. D.Schuch, Phys. Lett. A 338 (2005) 225-231
30. G. S. Agarwal and S. A. Kumar, Phys. Rev. Lett. 67 (1991) 3665
31. L. S.Brown, Phys. Rev. Lett. 66 (1991) 527

Renormalized solutions of nonlinear degenerated parabolic problems: Existence and uniqueness

Y. Akdim, J. Bennouna, M. Mekkour, and M. Rhoudaf

Département de Mathématiques Faculté des Sciences Dhar-Mahraz Fès, Morocco

In this paper, we study the existence and uniqueness of renormalized solutions for the nonlinear degenerated parabolic problem

$$\frac{\partial u}{\partial t} - \mathrm{div}(a(x,t,Du)) = f \quad \text{in} \;\; \Omega \times [0,T],$$
$$u(x,0) = u_0 \quad \text{in} \;\; \Omega$$
$$u = 0 \;\; \text{in} \;\; \partial\Omega \times [0,T]$$

where $a : \Omega \times [0,T] \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function satisfying the coercivity condition, the general growth condition and only the large monotonicity. The second term f belongs to $L^1(Q)$ and $u_0 \in L^1(\Omega)$.

Keywords: Weighted Sobolev spaces; Truncations; Renormalized solutions.

1. Introduction

Let Ω be a bounded open set of $\mathbb{R}^N$, p be a real number such that $2 < p < \infty$, $Q = \Omega \times [0,T]$ and $w = \{w_i(x),\ 0 \le i \le N\}$ be a vector of weight functions (i.e., every component $w_i(x)$ is a measurable function which is positive a.e. in Ω) satisfying some integrability conditions. The objective of this paper is to study the following problem, in the weighted Sobolev space,

$$\begin{aligned} \frac{\partial u}{\partial t} + Au = f \quad & \text{in } \Omega \times [0,T], \\ u = 0 \quad & \text{on } \partial\Omega \times]0,T[, \\ u(x,0) = u_0 \quad & \text{on } \Omega \end{aligned} \tag{1}$$

where $A = -\mathrm{div}(a(x,t,Du))$ and $a : \Omega \times [0,T] \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function, satisfying the coercivity condition

$$a(x,t,\xi).\xi \ge \alpha \sum_{i=1}^{N} w_i |\xi_i|^p,$$

the general growth condition

$$|a_i(x,t,\xi)| \leq \beta w_i^{1/p}(x)[k(x,t) + \sum_{j=1}^{N} w_j^{1/p'}(x)|\xi_j|^{p-1}],$$

and only the large monotonicity,

$$[a(x,t,\xi) - a(x,t,\eta)](\xi - \eta) \geq 0 \quad \text{for all } (\xi,\eta) \in \mathbb{R}^N \times \mathbb{R}^N,$$

f belongs to $L^1(Q)$.

The difficulties that arise in problem 1 are due to the following facts: the data f belongs to $L^1(Q)$ and a is monotone (but not necessarily strictly monotone).

In the classical Sobolev space, existence of a weak solution has been established by L. Boccardo and T. Gallouet,[2] and by D.Blanchard[3] when a is strictly monotone. We consider here renormalized solutions, for reason, this definition allows us to prove the uniqueness of a renormalized solution for problem 1, in contrast to the framework of weak solution where this question is still open. The notion of renormalized solution was introduced by J. Diperna and P.L. Lions[8] for the study of the Boltzmann equation.

For the parabolic equation (1) the existence and uniqueness of a renormalized solution has been proved by D. Blanchard and F. Murat[4] in the classical Sobolev space, they only supposed the large monotonicity. By L. Aharouch et al in[1] in the weighted Sobolev space with f belongs to the dual space and a is strictly monotone.

It is our purpose in this paper to generalize the result of[4] and prove the existence and uniqueness of renormalized solution for the problem (1.1) in the setting of the weighted Sobolev spaces. The proof uses techniques different from that given in [,[34]].

2. Preliminaries and basic assumptions

Let Ω be a bounded open set of $\mathbb{R}^N$, p be a real number such that $2 < p < \infty$ and $w = \{w_i(x),\ 0 \leq i \leq N\}$ be a vector of weight functions. Further, we suppose in all our considerations that , there exit

$$r_0 > \max(N,p) \text{ such that } w_i^{\frac{r_0}{r_0-p}} \in L^1_{\text{loc}}(\Omega), \tag{2}$$

$$w_i^{\frac{-1}{p-1}} \in L^1_{\text{loc}}(\Omega), \quad \text{for any } \ 0 \leq i \leq N. \tag{3}$$

$W^{1,p}(\Omega,w)$ is the space of all real-valued functions $u \in L^p(\Omega,w_0)$ such that the derivatives in the sense of distributions $\frac{\partial u}{\partial x_i} \in L^p(\Omega,w_i)$.

Which is a Banach space under the norm

$$\|u\|_{1,p,w} = \Big[\int_\Omega |u(x)|^p w_0(x)\,dx + \sum_{i=1}^N \int_\Omega |\frac{\partial u(x)}{\partial x_i}|^p w_i(x)\,dx\Big]^{1/p}. \quad (4)$$

The condition (3) implies that $C_0^\infty(\Omega)$ is a space of $W^{1,p}(\Omega,w)$ and consequently, we can introduce the subspace $V = W_0^{1,p}(\Omega,w)$ of $W^{1,p}(\Omega,w)$ as the closure of $C_0^\infty(\Omega)$ with respect to the norm (4). Moreover, condition (3) implies that $W^{1,p}(\Omega,w)$ as well as $W_0^{1,p}(\Omega,w)$ are reflexive Banach spaces.

Assumption (H1)

For $2 \le p < \infty$, we assume that the expression

$$|||u|||_V = \Big(\sum_{i=1}^N \int_\Omega |\frac{\partial u(x)}{\partial x_i}|^p w_i(x)\,dx\Big)^{1/p} \quad (5)$$

is a norm defined on V which equivalent to the norm (4), and there exist a weight function σ on Ω such that,

$$\sigma \in L^1(\Omega) \ \ and \ \ \sigma^{-1} \in L^1(\Omega).$$

We assume also the Hardy inequality,

$$\Big(\int_\Omega |u(x)|^q \sigma\,dx\Big)^{1/q} \le c\Big(\sum_{i=1}^N \int_\Omega |\frac{\partial u(x)}{\partial x_i}|^p w_i(x)\,dx\Big)^{1/p}, \quad (6)$$

holds for every $u \in V$ with a constant $c > 0$ independent of u, and moreover, the imbedding

$$W^{1,\,p}(\Omega,w) \hookrightarrow\hookrightarrow L^p(\Omega,\sigma), \quad (7)$$

expressed by the inequality (6) is compact. Note that $(V, |||.|||_V)$ is a uniformly convex (and thus reflexive) Banach space.

Assumption (H2)

$$|a_i(x,t,\xi)| \le \beta w_i^{1/p}(x)[k(t,x) + \sum_{j=1}^N w_j^{1/p'}(x)|\xi_j|^{p-1}] \ \ for \ \ 1 \le i \le N, \quad (8)$$

$$[a(x,t,\xi) - a(x,t,\eta)](\xi - \eta) \ge 0 \quad \text{for all } (\xi,\eta) \in \mathbb{R}^N \times \mathbb{R}^N, \quad (9)$$

$$a(x,t,\xi).\xi \ge \alpha \sum_{i=1}^N w_i|\xi_i|^p, \quad (10)$$

$$a(x,t,0) = 0 \quad (11)$$

where $k(x,t)$ is a positive function in $L^{p'}(Q)$, and α, β are strictly positive constants. We recall that, for $k > 1$ and s in $\mathbb{R}$, the truncation T_k is defined as,

$$T_k(s) = \begin{cases} s & \text{if } s \leq k \\ k\frac{s}{|s|} & \text{if } |s| > k. \end{cases}$$

And we define the function that will be used later $\varphi_k(r) = \int_0^r T_k(s)ds$.

3. Main results

Consider the problem

$$\begin{aligned} u_0 \in L^1(\Omega), \quad f \in L^1(Q) \\ \frac{\partial u}{\partial t} - \operatorname{div}(a(x,t,Du)) = f \quad \text{in } Q \\ u = 0 \quad \text{on } \partial\Omega \times]0,T[, \\ u(x,0) = u_0 \quad \text{on } \Omega. \end{aligned} \tag{12}$$

Definition 3.1. Let $f \in L^1(Q)$ and $u_0 \in L^1(\Omega)$. A real-valued function u defined on $\Omega \times [0,T]$ is a renormalized solution of problem (12) if:

$$u \in C([0,T]; L^1(\Omega)); \tag{13}$$

$$T_k(u) \in L^p(0,T; W_0^{1,\,p}(\Omega, w)) \quad for \quad all \quad (k \geq 0); \tag{14}$$

$$\forall\ c \geq 0, \quad T_{k+c}(u) - T_k(u) \to 0 \ \textit{strongly in}\ L^p(0,T; W_0^{1,\,p}(\Omega, w)) \ \textit{as}\ k \to +\infty \tag{15}$$

$$\frac{\partial S(u)}{\partial t} - div\,(S'(u)a(x,t,Du)) + S''(u)a(x,t,Du)Du = fS'(u) \quad in \quad D'(Q) \tag{16}$$

for all functions $S \in C^\infty(\mathbb{R})$ such that S' has a compact support in $\mathbb{R}$ (i.e $S' \in \ D(\mathbb{R})$),

$$u(t=0) = u_0. \tag{17}$$

Remark 1. Equation (16) is formally obtained by pointwise multiplication of equation (12) by $S'(u)$. Nevertheless, this process cannot be justified in general, so that weak solutions are not always renormalized solutions. In order terms, equation (16) is nothing but

$$-\int_Q S(u)\frac{\partial \varphi}{\partial t} + \int_Q S'(u)a(Du)D\varphi + \int_Q S''(u)a(Du)Du\varphi = \int_Q fS'(u)\varphi,$$

for all $\varphi \in C_0^\infty(Q)$ and all $S \in C^\infty(\mathbb{R})$ with $S' \in C_0^\infty(\mathbb{R})$. This formally corresponds to using in problem (12) the test function $\varphi S(u)$, but once again, this is only formal.
Indeed, if M is such that $suppS' \subset [-M, M]$, the following identifications are made in (16):
• $S(u)$ belongs to $L^\infty(Q)$ since S is a bounded function.
• $S'(u)a(x,t,Du)$ identifies with $S'(u)a(x,t,DT_M(u))$ a.e in Q. Since $T_M(u) \le M$ a.e in Q and $S'(u) \in L^\infty(Q)$, we obtain from (8) and (14) that

$$S'(u)a(x,t,DT_M(u)) \in \prod_{i=1}^{N} L^{p'}(Q, w_i^*)$$

• $S''(u)a(x,t,Du)Du$ identifies with $S''(u)a(DT_M(u))DT_M(u)$ and $S''(u)a(DT_M(u))DT_M(u) \in L^1(Q)$.
• $S'(u)f$ belongs to $L^1(Q)$.
The above considerations show that equation (16) holds in $D'(Q)$ and $\frac{\partial S(u)}{\partial t}$ belongs to $L^{p'}(0,T; W^{-1,\,p'}(\Omega, w_i^*)) + L^1(Q)$. It follows that u belongs to $C^0(0,T; L^1(\Omega))$ so that the initial condition (17) makes sense.

Theorem 3.1. *Let $f \in L^1(Q)$ and $u_0 \in L^1(\Omega)$. Assume that (H1) and (H2), there exists a unique renormalized solution u of problem* (12) .

Proof of the existence result
Step 1: The approximate problem
Consider the approximate problem:

$$\begin{gathered} u_n \in L^p(0,T; W_0^{1,\,p}(\Omega, w)) \\ \frac{\partial u_n}{\partial t} - \operatorname{div}(a(x,t,Du_n)) = f_n \quad in \quad D'(Q), \\ u_n(t=0) = u_{0n}, \end{gathered} \tag{18}$$

with (f_n) be a sequence of smooth functions such that $f_n = T_n(f)$ and $f_n \to f \quad in \quad L^1(Q)$, and (u_{0n}) be a sequence such that $u_{0n} = T_n(u_0)$ and $u_{0n} \to \quad u_0 \quad in \quad L^1(\Omega)$. f_n and u_{0n} belong, respectively to $L^\infty(Q)$ and $L^\infty(\Omega)$.
Let $X = L^p(0,T; W_0^{1,\,p}(\Omega, w))$, $X^* = L^{p'}(0,T; W^{-1,p'}(\Omega, w^*))$, $H = L^2(\Omega, \sigma)$ and $W_p^1(0,T,V,H) = \{v \in X : v' \in X^*\}$.
The operator A is bounded, demicontinuous, pseudomonone with respect to $D(L)$ where $D(L) = \{v \in X; v' \in X^*, \ \ v(0) = 0\}$ and strongly coercive. So all conditions of theorem 5 in[5] are met. Therefore, there exists a solution

$u_n \in D(L)$ of the evolution equation (18) for any $f_n \in X^*$. By the definition of $D(L)$ and see,[1],[10] we obtain

$$D(L) \subseteq W_p^1(0,T,V,H) \subseteq C([0,T],H).$$

Which implies that $u_n \in C([0,T],H)$.

Step 2: Some estimations about the truncated sequence of solutions.

Using in (18) the test function $T_k(u_n)\chi_{(0,\tau)}$, we get, for every $\tau \in [0,T]$ (Since $|\varphi_k(r)| \leq k\,|r|$).

$$\int_Q \frac{\partial u_n}{\partial t} T_k(u_n)dx \geq \int_\Omega \varphi_k(u_n(\tau))dx - k\,\|u_{0n}\|_{L^1(\Omega)}\,.$$

Then we deduce that,

$$\begin{array}{l} \int_\Omega \varphi_k(u_n(\tau))dx + \int_0^\tau \int_\Omega a(x,t,Du_n)DT_k(u_n)dxdt \\ \leq k(\|u_{0n}\|_{L^1(\Omega)} + \|f_n\|_{L^1(Q)}) \leq ck. \end{array}$$

Thanks to (10) and by the fact that $\varphi_k(u_n(\tau)) \geq 0$, we deduce that,

$$\begin{array}{l} \alpha \int_Q \sum\limits_{i=1}^N w_i(x) \left|\frac{\partial T_k(u_n)}{\partial x_i}\right|^p dxdt \\ \leq ck, \quad \forall k \geq 1. \end{array} \tag{19}$$

Then, $T_k(u_n)$ is bounded in $L^p(0,T;W_0^{1,\,p}(\Omega,w))$, and $T_k(u_n) \rightharpoonup v_k$ in $L^p(0,T;W_0^{1,\,p}(\Omega,w))$. Using the compact imbedding (3.3) we get,

$$T_k(u_n) \to v_k \quad strongly \quad in \quad L^p(Q,\sigma) \quad and \quad a.e \quad in \quad Q.$$

Let $k>0$ large enough and B_R be a ball of Ω, we have,

$$\begin{array}{l} k \;\; meas(\{|u_n|>k\} \cap B_R \times [0,T]) = \int_0^T \int_{\{|u_n|>k\}\cap B_R} |T_k(u_n)|\,dxdt \\ \leq \int_0^T \int_{B_R} |T_k(u_n)|\,dxdt \end{array}$$

$$\leq Tc_R \left(\int_Q \sum_{i=1}^N w_i(x) \left| \frac{\partial T_k(u_n)}{\partial x_i} \right|^p dxdt \right)^{\frac{1}{p}} \leq ck^{\frac{1}{p}}$$

So, we have

$$\lim_{k\to+\infty} (meas(\{|u_n|>k\} \cap B_R \times [0,T])) = 0.$$

Consider now a function non decreasing $g_k \in C^2(\mathbb{R})$ such that $g_k(s) = s \;\; for \;\; |s| \leq \frac{k}{2}$ and $g_k(s) = k \;\; for \;\; |s| \geq k$

Multiplying the approximate equation by $g_k'(u_n)$, we get

$$\frac{\partial g_k(u_n)}{\partial t} - div(a(x,t,Du_n)g_k'(u_n)) + a(x,t,Du_n)g_k''(u_n) = f_n g_k'(u_n)$$

in the sense of distributions. This implies, thanks to the fact g_k' has compact support, that $g_k(u_n)$ is bounded in X, while it's time derivative $\frac{\partial g_k(u_n)}{\partial t}$ is bounded in $X^* + L^1(Q)$.
Hence see Lemma 3.8 in[1] allows us to conclude that $g_k(u_n)$ is compact in $L^p_{loc}(Q,\sigma)$. Thus, for a subsequence, it also converges in measure and almost every where in Q (since we have, for every $\lambda > 0$,)

$$meas(\{|u_n - u_m| > \lambda\} \cap B_R \times [0,T]) \leq meas(\{|u_n| > k\} \cap B_R \times [0,T])$$

$$+meas(\{|u_m| > k\} \cap B_R \times [0,T]) + meas(\{|g_k(u_n) - g_k(u_m)| > \lambda\})$$

Let $\varepsilon > 0$, then, there exist $k(\varepsilon) > 0$ such that,

$$meas(\{|u_n - u_m| > \lambda\} \cap B_R \times [0,T]) \leq \varepsilon \quad for \quad all \quad n,m \geq n_0(k(\varepsilon),\lambda,R).$$

This proves that (u_n) is a Cauchy sequence in measure in $B_R \times [0,T]$, thus converges almost everywhere to some measurable function u. Then for a subsequence denoted again u_n, we can deduce from (19) that,

$$T_k(u_n) \rightharpoonup T_k(u) \quad weakly \quad in \quad L^p(0,T;W_0^{1,\,p}(\Omega,w)) \tag{20}$$

and then, the compact imbedding (3.3) gives,

$$T_k(u_n) \to T_k(u) \quad strongly \quad in \quad L^p(Q,\sigma) \quad and \quad a.e \quad in \quad Q.$$

Which implies, by using (8), for all $k > 0$ that there exists a function $h_k \in \prod_{i=1}^{N} L^{p'}(Q,w_i^*)$, such that

$$a(x,t,DT_k(u_n)) \rightharpoonup h_k \quad weakly \quad in \quad \prod_{i=1}^{N} L^{p'}(Q,w_i^*) \tag{21}$$

We now establish that u belongs to $L^\infty(0,T;L^1(\Omega))$. Using (19), (20) and passing to the limit-inf as n tends to $+\infty$ and we obtain

$$\int_\Omega \varphi_k(u)(t)dx \leq k[\|f\|_{L^1(\Omega)} + \|u_0\|_{L^1(\Omega)}]$$

By the definition of φ_k, we deduce from the above inequality that

$$k\int_\Omega |u(x,t)|\,dx \leq \frac{3k^2}{2} meas(\Omega) + k[\|f\|_{L^1(\Omega)} + \|u_0\|_{L^1(\Omega)}]$$

for almost any t in (0,T), which gives that u belong to $L^\infty(0,T;L^1(\Omega))$.
Step 3: In this step we identify the weak h_k in (21) and we prove the

weak-L^1 convergence of the "*truncated*" energy $a(x,t,DT_k(u_n))DT_k(u_n)$ as n *tends to* $+\infty$. We can prove easily the limits

$$\lim_{m\to+\infty}\limsup_{n\to+\infty}\int_{\{m\le|u_n|\le m+1\}} a(Du_n)Du_n dxdt = 0. \tag{22}$$

We define for all $\mu \ge 0$ and all $(x,t)\in Q$,

$$v_\mu = \mu\int_\infty^t \tilde{v}(x,s)\exp(\mu(s-t))ds \quad where \quad \tilde{v}(x,s) = v(x,s)\chi_{(0,T)}(s).$$

See proposition 3.1-3.3 in.[1]

Lemma 3.1. *Let $k \ge 0$ be fixed. Let $(T_k(u))_\mu$ the mollification of $T_k(u)$. Let S be an increasing $C^\infty(\mathbb{R})$-function such that $S(r) = r$ for $|r| \le k$ and supp S' is compact. Then,*

$$\lim_{\mu\to+\infty}\lim_{n\to+\infty}\int_0^T\int_0^t\left\langle\frac{\partial S(u_n)}{\partial t},(T_k(u_n)-(T_k(u))_\mu)\right\rangle dtds \ge 0, \tag{23}$$

where $\langle .,.\rangle$ denotes the duality pairing between $L^1(\Omega)+W^{-1,p'}(\Omega,w^*)$ and $L^\infty(\Omega)\cap W_0^{1,p}(\Omega,w)$.

Proof of Lemma 3.1. Let $k \ge 0$ be fixed. Let us first recall that since supp S' is compact, we have

$$S(u_n)\in X\cap L^\infty(Q),\quad S(u)\in X\cap L^\infty(Q) \quad and \quad \frac{\partial S(u_n)}{\partial t}\in L^1(Q)+X^*.$$

Moreover, since S is increasing and $S(r) = r$ *for* $|r| \le k$,

$$T_k(S(u_n)) = T_k(u_n) \quad and \quad T_k(S(u)) = T_k(u) \quad a.e. \ in \ Q.$$

As a consequence, $(T_k(S(u)))_\mu = (T_k(u))_\mu$ *a.e. in* Q for any $\mu > 0$. It follows that under the notation $v_n = S(u_n)$ *and* $v = S(u)$, we have

$$\begin{aligned}
&\int_0^T\int_0^t\left\langle\frac{\partial S(u_n)}{\partial t},(T_k(u_n)-(T_k(u))_\mu)\right\rangle dtds\\
&=\int_0^T\int_0^t\left\langle\frac{\partial (v_n)}{\partial t},(T_k(v_n)-(T_k(v))_\mu)\right\rangle dtds\\
&=\int_0^T\int_0^t\left\langle\frac{\partial (v_n-(T_k(v))_\mu)}{\partial t},(v_n-(T_k(v))_\mu)\right\rangle dtds\\
&-\int_0^T\int_0^t\left\langle\frac{\partial (v_n)}{\partial t},(v_n-T_k(v_n))\right\rangle dtds\\
&+\int_0^T\int_0^t\int_\Omega\frac{\partial (T_k(v))_\mu}{\partial t}(v_n-(T_k(v))_\mu)dxdtds,
\end{aligned}$$

$$= \frac{1}{2}\int_0^T \int_\Omega |(v_n - (T_k(v))_\mu)|^2 \, dxdt - \frac{T}{2}\int_\Omega |(v_n - (T_k(v))_\mu)|^2 \, (t=0)dx$$
$$- \frac{1}{2}\int_0^T \int_\Omega |v_n - (T_k(v_n))|^2 \, dxdt + \frac{T}{2}\int_\Omega |(v_n - (T_k(v_n)))|^2 \, (t=0)dx$$
$$+ \int_0^T \int_0^t \int_\Omega \frac{\partial (T_k(v))_\mu}{\partial t}(v_n - (T_k(v))_\mu)dxdtds, \tag{24}$$

since $\int_0^r (s - T_k(s))ds = \frac{1}{2}|r - T_k(r)|^2$.
To pass to the limit in (24) as n tends to $+\infty$, we first observe that since S is bounded, the definition of v_n and v together with convergence of u_n a.e in Q imply that

$$v_n \to v \quad strongly \quad in \quad L^2(Q) \quad and \quad in \quad L^\infty weak - *.$$

Moreover, the initial condition for u_n gives

$$v_n(t=0) = S(u_n)(t=0) = S(u_{n0}) \quad a.e \quad in \quad Q,$$

so that the strong convergence of u_{0n} to u_0 in $L^1(\Omega)$ implies that

$$v_n(t=0) \to S(u_0) \quad in \quad L^2(\Omega).$$

Passing to the limit as n tends to $+\infty$ in (24) is an easy task and leads to

$$\lim_{n\to+\infty} \int_0^T \int_0^t \left\langle \frac{\partial S(u_n)}{\partial t}, (T_k(u_n) - (T_k(u))_\mu) \right\rangle dtds$$
$$= \frac{1}{2}\int_0^T \int_\Omega |(v - (T_k(v))_\mu)|^2 \, dxdt - \frac{T}{2}\int_\Omega |(S(u_0) - (T_k(v))_\mu)|^2 \, (t=0)dx$$
$$- \frac{1}{2}\int_0^T \int_\Omega |v - (T_k(v))|^2 \, dxdt + \frac{T}{2}\int_\Omega |S(u_0) - T_k(S(u_0))|^2 \, dx$$
$$+ \int_0^T \int_0^t \int_\Omega \frac{\partial (T_k(v))_\mu}{\partial t}(v - (T_k(v))_\mu)dxdtds, \tag{25}$$

for any $\mu > 0$.
In order to pass to the limit-inf as μ tends to infinity in (25), we now use the definition of $(T_k(u))_\mu$, in terms of $T_k(v)$, rewrite as

$$\frac{\partial (T_k(v))_\mu}{\partial t} + \mu((T_k(v))_\mu - T_k(v)) = 0 \quad in \quad D'(Q), \tag{26}$$

$$(T_k(v))_\mu(t=0) = v_0^\mu \quad in \quad \Omega, \tag{27}$$

$$(T_k(v))_\mu \to T_k(v) \quad in \quad L^2(Q) \tag{28}$$

$$(T_k(v))_\mu(t=0) \to T_k(S(u_0)) \quad in \quad L^2(Q) \tag{29}$$

as μ tends to $+\infty$, since again $T_k(S(u_0)) = T_k(u_0)$ $a.e$ in Ω.
In view of (26), (28), (29) passing to the limit-inf as μ tends to $+\infty$ in (25) leads to

$$\begin{aligned} &\liminf_{\mu\to+\infty} \lim_{n\to+\infty} \int_0^T \int_0^t \left\langle \frac{\partial S(u_n)}{\partial t}, (T_k(u_n) - (T_k(u))_\mu)\right\rangle dtds \\ &= \liminf_{\mu\to+\infty} \mu \int_0^T \int_0^t \int_\Omega (T_k(v) - (T_k(v))_\mu)(v - (T_k(v))_\mu) dxdtds. \end{aligned} \tag{30}$$

The proof of lemma 3.1 then reduces to show that the right-hand side of (30) is nonnegative. To this end we just remark that

$$(T_k(v) - (T_k(v))_\mu)(v - (T_k(v))_\mu) \geq 0 \quad a.e \quad in \quad Q.$$

We prove the following lemma, which is the key point in the monotonicity arguments that will be developed in undo lemma .

Lemma 3.2. *The subsequence of u_n satisfies for any $k \geq 0$*

$$\limsup_{n\to+\infty} \int_0^T \int_0^t \int_\Omega a(DT_k(u_n))DT_k(u_n) dxdtds \leq \int_0^T \int_0^t \int_\Omega h_k DT_k(u) dxdtds, \tag{31}$$

where h_k is defined in (21).

In the following we adapt the above-mentioned method to problem (12) and we first introduce a sequence of increasing $C^\infty(\mathbb{R})$-functions S_m such that

$$S_m(r) = r \quad if \quad |r| \leq m,$$

$$suppS'_m \subset [-(m+1), m+1],$$

$$\|S''_m\|_{L^\infty} \leq 1, \quad for \quad any \quad m \geq 1.$$

Pointwise multiplication of equation (18) by $S'_m(u_n)$ (which is licit) leads to

$$\begin{aligned} &\tfrac{\partial S_m(u_n)}{\partial t} - div\,(S'_m(u_n)a(x,t,Du_n)) \\ &+S''_m(u_n)a(x,t,Du_n)Du_n = f_n S'(u_n) \quad in \quad D'(Q) \end{aligned} \tag{32}$$

which is nothing but the renormalized of formulation (16) for u_n with $S = S_m$. Remark that (32) implies that $\frac{\partial S_m(u)}{\partial t} \in L^1(Q) + X^*$.

The use of the test function $W_\mu^n = T_k(u_n) - (T_k(u))_\mu, \quad k \geq 0$ in equation.(32) we obtain upon integration over (0,t) and then over (0,T):

$$\int_0^T \int_0^t \left\langle \frac{\partial S_m(u_n)}{\partial t}, W_\mu^n \right\rangle dtds + \int_0^T \int_0^t \int_\Omega S'_m(u_n) a(Du_n) DW_\mu^n$$
$$+ \int_0^T \int_0^t \int_\Omega S''_m(u_n) a(Du_n) Du_n W_\mu^n dxdtds = \int_0^T \int_0^t \int_\Omega f_n S'_m(u_n) W_\mu^n. \tag{33}$$

We pass to the limit in (33) as n tends to $+\infty$, μ tends to $+\infty$ and then m tends to $+\infty$, the real number $k \geq 0$ being fixed. In order to perform this task we prove below the following results for fixed $k \geq 0$:

$$\liminf_{\mu \to +\infty} \lim_{n \to +\infty} \int_0^T \int_0^t \left\langle \frac{\partial S_m(u_n)}{\partial t}, W_\mu^n \right\rangle dtds \geq 0, \quad for \quad any \quad m \geq k, \tag{34}$$

$$\lim_{m \to +\infty} \limsup_{\mu \to +\infty} \limsup_{n \to +\infty} \left| \int_0^T \int_0^t \int_\Omega S''_m(u_n) a(Du_n) Du_n W_\mu^n dxdtds \right| = 0, \quad m \geq 1 \tag{35}$$

$$\lim_{\mu \to +\infty} \lim_{n \to +\infty} \int_0^T \int_0^t \int_\Omega f_n S'_m(u_n) W_\mu^n dxdtds = 0 \quad for \quad any \quad m \geq 1 \tag{36}$$

Proof of (34). The function S_m belongs to $C^\infty(\mathbb{R})$ and is increasing. We have for $m \geq k$, $S_m(r) = r \quad for \quad |r| \leq k$ while $suppS'_m$ is compact.
In view of the definition of W_μ^n, lemma 3.1 applies with $S = S_m$ for fixed $m \geq k$. As a consequence (34) holds true.
Proof of (35) In order to avoid repetitions in the proofs of (36), let us summarize the properties of W_μ^n. For fixed $\mu > 0$

$$W_\mu^n \rightharpoonup T_k(u) - (T_k(u))_\mu \quad weakly \quad in \quad L^p(0,T; W_0^{1,\,p}(\Omega, w)), \quad as \quad n \to +\infty$$

$$\left\| W_\mu^n \right\|_{L^\infty(Q)} \leq 2k, \quad for \quad any \quad n > 0 \quad and \quad for \quad any \quad \mu > 0$$

we deduce that for fixed $\mu > 0$

$$W_\mu^n \to T_k(u) - (T_k(u))_\mu \quad a.e \quad in \quad Q \quad and \quad in \quad L^\infty(Q) weak-*, \quad as \quad n \to +\infty$$

one has $suppS''_m \subset [-(m+1), -m] \cup [m, m+1]$ for any $m \geq 1$. As a consequence

$$\left| \int_0^T \int_0^t \int_\Omega S''_m(u_n) a(Du_n) Du_n W_\mu^n \right|$$
$$\leq T \left\| S''_m(u_n) \right\|_{L^\infty} \left\| W_\mu^n \right\|_{L^\infty} \int_{\{m \leq |u_n| \leq m+1\}} a(Du_n) Du_n$$

for any $m \geq 1$, *any* $\mu > 0$ *and any* $n \geq 1$
it possible to obtain

$$\limsup_{\mu\to+\infty}\limsup_{n\to+\infty}\left|\int_0^T\int_0^t\int_\Omega S''_m(u_n)a(Du_n)Du_nW_\mu^n dxdtds\right|$$

$$\leq C\limsup_{n\to+\infty}\int_{\{m\leq|u_n|\leq m+1\}} a(Du_n)Du_n dxdt,$$

for any $m \geq 1$, where C is a constant independent of m. Appealing now to (22) it possible to pass the limit as m tends to $+\infty$ to establish (35).
Proof of (36) Lebesque's convergence theorem implies that for any $\mu > 0$ and any $m \geq 1$

$$\lim_{n\to+\infty}\int_0^T\int_0^t\int_\Omega f_nS'_m(u_n)W_\mu^n dxdtds$$
$$=\int_0^T\int_0^t\int_\Omega fS'_m(u)(T_k(u)-(T_k(u)_\mu))dxdtds$$

Now, for fixed $m \geq 1$, using[1] makes it possible to pass to the limit as $\mu \to +\infty$ in the above equality to obtain (36).
We now turn back to the proof of lemma 3.2. Due to (34), (35) and (36), we are in a position to pass the limit-sup when n tends to $+\infty$, then to the limit-sup when μ tends $+\infty$ and then to the limit as m tends to $+\infty$ in (33). Using the definition of W_μ^n we obtain that for any $k \geq 0$

$$\lim_{m\to+\infty}\limsup_{\mu\to+\infty}\limsup_{n\to+\infty}\int_0^T\int_0^t\int_\Omega S'_m(u_n)a(Du_n)$$
$$(DT_k(u_n)-D(T_k(u))_\mu)dxdtds \leq 0.$$

Since for $k \leq n$ and $k \leq m$, the above inequality implies that for $k \leq m$

$$\limsup_{n\to+\infty}\int_0^T\int_0^t\int_\Omega a(Du_n)DT_k(u_n)dxdtds$$
$$\leq \lim_{m\to+\infty}\limsup_{\mu\to+\infty}\limsup_{n\to+\infty}\int_0^T\int_0^t\int_\Omega S'_m(u_n)a(Du_n)D(T_k(u))_\mu dxdtds \tag{37}$$

The right-hand side of (37) is computed as follows. We have for $n \geq m+1$:

$$S'_m(u_n)a(Du_n) = S'_m(u_n)a(DT_{m+1}(u_n)) \quad a.e \quad in\ Q.$$

Due to the weak convergence of $a(DT_{m+1}(u_n))$ it follows that for fixed $m \geq 1$

$$S'_m(u_n)a(Du_n) \rightharpoonup S'm(u_n)h_{m+1} \quad weakly \quad in \quad \prod_{i=1}^N L^{p'}(Q,w_i^*)$$

when n tends to $+\infty$. The strong convergence of $(T_k(u))_\mu$ *to* $T_k(u)$ in $L^p(0,T;W_0^{1,\,p}(\Omega,w))$ as μ tends to $+\infty$, then we have

$$\lim_{\mu\to+\infty}\lim_{n\to+\infty}\int_0^T\int_0^t\int_\Omega S'_m(u_n)a(Du_n)D(T_k(u))_\mu = \int_0^T\int_0^t\int_\Omega S'_m(u_n)h_{m+1}DT_k(u), \tag{38}$$

as soon as $k\le m$, $S'm(r)=1$ *for* $|r|\le m$. Now for $k\le m$ we have,

$$a(DT_{m+1}(u_n))\chi_{\{|u_n|<k\}} = a(DT_k(u_n))\chi_{\{|u_n|<k\}}\quad a.e\ \ in\ \ Q,$$

which implies that, passing to the limit as $n\to+\infty$,

$$h_{m+1}\chi_{\{|u_n|<k\}} = h_k\chi_{\{|u_n|<k\}}\quad a.e\ \ in\ \ Q-\{|u|=k\}\ \ for\ \ k\le m. \tag{39}$$

As a consequence of (39) we have for $k\le m$,

$$h_{m+1}DT_k(u) = h_kDT_k(u)\quad a.e\ \ in\ \ Q. \tag{40}$$

Using (37), (38), (40) we obtain (31) and the proof of lemma 3.2 is complete. Let $k\ge 0$ be fixed. Using (9) we have

$$\lim_{n\to+\infty}\int_0^T\int_0^t\int_\Omega[a(DT_k(u_n))-a(DT_k(u))][DT_k(u_n)-DT_k(u)]dxdtds\ge 0. \tag{41}$$

By lemma 3.2, weak convergence of $T_k(u_n)$ and $a(DT_k(u_n))$ we pass to the limit-sup as $n\to+\infty$ in (41) and obtain the result :

$$\lim_{n\to+\infty}\int_0^T\int_0^t\int_\Omega[a(DT_k(u_n))-a(DT_k(u))][DT_k(u_n)-DT_k(u)]dxdtds=0. \tag{42}$$

Lemma 3.3. *For fixed* $k\ge 0$, *we have*

$$h_k = a(DT_k(u))\quad a.e\ \ in\ \ Q, \tag{43}$$

$$a(DT_k(u_n))DT(u_n)\rightharpoonup a(DT_k(u))DT_k(u)\quad weakly\ \ in\ \ L^1(Q). \tag{44}$$

Proof of Lemma 3.3. The proof is standard, from weak convergence of $(T_k(u_n))$ and $a(DT_k(u_n))$, (42) and (9) we deduce the result.

Step 4: Convergence of (u_n) in $C([0,T],L^1(\Omega))$

Proposition 3.1. *The sequence* (u_n) *is a Cauchy sequence in* $C([0,T],L^1(\Omega))$.

Moreover, $u \in C([0,T], L^1(\Omega))$ *and* u_n *converges to* u *in* $C([0,T], L^1(\Omega))$.

The proof is similar of Proposition 7.3.1 in.[9]

Step 5: In this step we prove that u satisfies (15). Using in (18) the admissible test function $T_{k+c}(u_n) - T_k(u_n)$ leads to

$$\begin{aligned}&\int_\Omega [\varphi_{k+c}(u_n)(T) - \varphi_k(u_n)(T)]dx + \int_Q a(Du_n)[DT_{k+c}(u_n) - DT_k(u_n)]dxdt \\ &= \int_Q f_n[T_{k+c}(u_n) - T_k(u_n)]dxdt + \int_\Omega [\varphi_{k+c}(u_{0n}) - \varphi_k(u_{0n})]dx.\end{aligned} \tag{45}$$

In (45) and in what follows the (t, x) dependence on $a(x, t, d)$ is dropped for convenience. Since the function $\varphi_{k+c}(r) - \varphi_k(r)$ is positive, using the convergence of f_n to f in $L^1(Q)$, the convergence of u_{0n} to u_0 in $L^1(\Omega)$, the fact that $|T_{k+c}(r) - T_k(r)| \leq C$, the linear growth of φ_k at infinity, and (3.1), we deduce from (45)that:

$$\begin{aligned}&\limsup_{n\to\infty} \int_Q a(Du_n)[DT_{k+c}(u_n) - DT_k(u_n)]dxdt \\ &\leq \int_Q f[T_{k+c}(u) - T_k(u)]dxdt + \int_\Omega [\varphi_{k+c}(u_0) - \varphi_k(u_0)]dx.\end{aligned} \tag{46}$$

Since $DT_{k+c}(u_n) - DT_k(u_n) = 0$ when (t, x) does not belong to the set $\{(t, x);\ k \leq |u_n(t, x)| \leq k + c\}$, we have almost pointwise in Q

$$\begin{aligned}&a(Du_n)[DT_{k+c}(u_n) - DT_k(u_n)] \\ &= a(DT_{k+c}(u_n) - DT_k(u_n))[DT_{k+c}(u_n) - DT_k(u_n)]\end{aligned} \tag{47}$$

Since, by (20), $T_{k+c}(u_n) - T_k(u_n)$ weakly converge to $T_{k+c}(u) - T_k(u)$ in X as n tends to ∞, we obtain

$$\begin{aligned}&\alpha \limsup_{n\to\infty} \int_Q \sum_{i=1}^N \left|\frac{\partial[T_{k+c}(u) - T_k(u)]}{\partial x_i}\right|^p w_i(x)dxdt \\ &\leq \int_Q f[T_{k+c}(u) - T_k(u)]dxdt + \int_\Omega [\varphi_{k+c}(u_0) - \varphi_k(u_0)]dx.\end{aligned} \tag{48}$$

When k tends to infinity, the right-hand side (48) tends to zero: indeed $T_{k+c}(u) - T_k(u)$ is dominated by C and tends almost everywhere to zero, while

$$\left|\int_\Omega [\varphi_{k+c}(u_0) - \varphi_k(u_0)]\right| \leq C \int_{\{|u_0(x)|>k\}} |u_0(x)|\, dx.$$

This proves that u satisfies (15).

Step 6: Passing to the limits. We now return to the proof of the existence result of theorem (3.1). We notice that the function u defined in Lemma 20 meets the regularity requirements (15), (13) and (16) of Definition 3.1. The initial condition is easily recovered by passing to the limit in

$$u_n(t=0) = u_{0n},$$

and using (3.1). It thus only remains to prove that u satisfies equation (16). Since u_n belongs to $L^p(0,T;W_0^{1,\,p}(\Omega,w))$ with $\frac{\partial u_n}{\partial t}$ in $L^{p'}(0,T;W^{-1,p'}(\Omega,w^*))$, we have for any $S \in C^\infty(\mathbb{R})$ with $S' \in C_0^\infty(\mathbb{R})$

$$\frac{\partial S(u_n)}{\partial t} = S'(u_n)\frac{\partial u_n}{\partial t} \;\; in \;\; D'(Q).$$

Using this fact in equation (18) leads to:

$$\frac{\partial S(u_n)}{\partial t} - div[S'(u_n)a(Du_n)] + S''(u_n)a(Du_n)Du_n = fS'(u_n) \;\; in \;\; D'(Q). \tag{49}$$

To pass to the limit in (49) as n tends to $+\infty$, let M be such that $supp\ S' \subset [-M,M]$, so that

$$S'(u_n)a(Du_n) = S'(u_n)a(DT_M(u_n)).$$

The pointwise convergence of $u_n \;\; to \;\; u$ as n tends to $+\infty$ and the bounded character of S' permit us to conclude that

$$S'(u_n)a(Du_n) \rightharpoonup S'(u)a(DT_M(u)) \;\; in \;\; \prod_{i=1}^{N} L^{p'}(Q,w_i^*). \tag{50}$$

as n tends to $+\infty$. $S'(u)a(DT_M(u))$ has been denoted by $S'(u)a(Du)$ in equation (16).

As far as the 'energy' term

$$S''(u_n)a(Du_n)Du_n = S''(u_n)a(DT_M(u_n))DT_M(u_n)$$

is concerned, remark that

$$\lim_{n\to+\infty}\int_Q [a(DT_M(u_n)) - a(DT_M(u))][DT_M(u_n) - DT_M(u)]dxdt = 0.$$

Due to the monotonicity (9), the above equality shows that

$$[a(DT_M(u_n)) - a(DT_M(u))][DT_M(u_n) - DT_M(u)] \to 0 \;\; strongly \;\; in \;\; L^1(Q) \tag{51}$$

as n tends to $+\infty$. Again using (51), (20) and (44) then implies that

$$a(DT_M(u_n))DT_M(u_n) \rightharpoonup a(DT_M(u))DT_M(u) \quad weakly \quad in \quad L^1(Q) \quad (52)$$

as n tends to $+\infty$. Appealing now to the pointwise convergence of $S''(u_n)$ to $S''(u)$ and since S'' is bounded, a straightforward application of Egoroff's theorem leads to

$$S''(u_n)a(DT_M(u_n))DT_M(u_n) \rightharpoonup S''(u)a(DT_M(u))DT_M(u)$$
$$weakly \ in \ L^1(Q), \quad (53)$$

as n tends to $+\infty$.
Finally, the pointwise convergence (3.1) and the $L^1(Q)-$strong convergence of f_n to f yield

$$S'(u_n)f_n \to S'(u)f \quad strongly \quad in \quad L^1(Q), \quad (54)$$

as n tends to $+\infty$. The $L^\infty-$weak* convergence of $S(u_n)$ to $S(u)$ together with (49), (50), (53) and (54) proves that (16) is satisfied in the sense of distributions. This completes the proof of the existence result of Theorem 3.1.

Proof of uniqueness result

We now turn to the proof of the uniqueness result of Theorem 3.1. In fact, we will prove a comparison result for renormalised solutions.

Lemma 3.4. *Assume that u_{01} and u_{02} lie in $L^1(\Omega)$, that f_1 and f_2 lie in $L^1(Q)$ and that they satisfy:*

$$u_{01} \leq u_{02}, \quad (55)$$

$$f_1 \leq f_2. \quad (56)$$

Then if u_1 and u_2 are two renormalised solutions of problem (18) respectively for the data (f_1, u_{01}) and (f_2, u_{02}), we have

$$u_1 \leq u_2 \quad almost \quad everywhere \quad in \quad Q. \quad (57)$$

Remark 2. Uniqueness of a solution of (13)-(16) is a direct application of Lemma 3.4.

References

1. L.Aharouch, E.Azroul and M.Rhoudaf, *Strongly nonlinear variational parabolic problems in weighted sobolev spaces.* The Australian journal of Mathematical Analysis and Applications, Vol.5, Issue 2, Article 13, pp. 1-25,2008 .
2. L. Boccardo and T.Gallouet. On some nonlinear elliptic and parabolic equations involving measure data.J.Funct.Anal.87(1989),149-69.
3. D. Blanchard. Truncations and monotonocity methods for parabolic equations. Nonlinear Anal.21 (1993),725-43.
4. D. Blanchard, F.Murat. Renormalized solutions of nonlinear parabolic problems with L^1 data: existence and uniqueness. Proceedings of the Royal Society of Edinburgh, 127A, 1137-1152, 1997.
5. J.Berkovits and V.Mustonen. Topological degree for perturbations of linear maximal monotone mappings and applications to a class of parabolic problem . Rendiconti di Matematica, Serie VII, volume 12, Roma (1992),597-621.
6. P. Drabek, A. Kufner and F. Nicolosi, *Non linear elliptic equations, singular and degenerated cases*, University of West Bohemia, (1996).
7. J. L. Lions *quelques méthodes de résolution des problème aux limites non lineaires*, Dundo, Paris (1969).
8. P. L. Lions and R.J.Diperna, *On the cauchy problem for Boltzman equations: global existence and weak stability. Ann.of Math.(2)130*(1989),321-366.
9. M.Rhoudaf, *Etude de certains problemes elliptiques et paraboliques non lineaires avec second membre L^1 ou dans le dual ,* Thèse de doctorat d' etat, Faculté des Sciences Dhar-Mhraz, Fès, Maroc, (2006) .
10. E.Zeidler , *Nonlinear Functional Analysis and its Applications, Springer-Verlag(New York- Heidlberg, (1990)).*

Solvability of quasilinear degenerated elliptic equations with L^1 data

Youssef Akdim

Laboratoire d'Informatique Mathématique Automatique et Optoeléctronique, Faculté Poly-disciplinaire, Taza, Morocco
E-mail: akdimyoussef@yahoo.fr

In this paper we prove the existence of solutions for quasilinear degenerated elliptic operators $A(u) + g(x, u, \nabla u) = f$, where A is a Leray-Lions operator from $W_0^{1,p}(\Omega, w)$ into its dual, while $g(x, s, \xi)$ is a nonlinear term which has a growth condition with respect to ξ and no growth with respect to s, but it satisfies a sign condition on s. The right hand side f is assumed to belong to $L^1(\Omega)$.

Keywords: Quasilinear degenerate elliptic operator; Existence result.

1. Introduction

In this paper, we shall be concerned with the existence of solutions for quasilinear degenerated elliptic equations of the type

$$(\mathcal{P}) \begin{cases} Au + g(x, u, \nabla u) = f \quad \text{in } \mathcal{D}'(\Omega), \\ u \in W_0^{1,p}(\Omega, w), \quad g(x, u, \nabla u) \in L^1(\Omega), \end{cases}$$

where

$$Au = -\text{div}(a(x, u, \nabla u))$$

is a weighted Leray-Lions operator from the weighted Sobolev space $X = W_0^{1,p}(\Omega, w)$ into $X^* = W^{-1,p'}(\Omega, w^*)$ and where g is a nonlinear lower order term having natural growth ($|g(x, s, \xi)| \leq b(|s|)(c(x) + \sum_{i=1}^{N} w_i|\xi_i|^p)$) which satisfies the sign condition $g(x, s, \xi)s \geq 0$. We also assume that $g(x, s, \xi)$ having an "exact natural growth" i.e., $|g(x, s, \xi)| \geq \rho \sum_{i=1}^{N} w_i|\xi_i|^p$. The right hand side f is assumed to belong to $L^1(\Omega)$. It will turn out that for any solution u, $g(x, u, \nabla u) \in L^1(\Omega)$, but for a general $v \in X$, $g(x, v, \nabla v)$ can be more singular.

Drabek and Nicolosi in[9] proved the existence of bounded solution for the degenerated problem $(\mathcal{P})$ where $g(x, u, \nabla u) = -c_0|u|^{p-2}u$, more precisely for the problem,

$$Au - c_0|u|^{p-2}u = f(x, u, \nabla u)$$

with some more general degeneracy, but under some other assumptions on f and $a(x, s, \xi)$. The existence result for the problem $(\mathcal{P})$ (respectively, unilateral problem) where f lies in the dual space $W^{-1,p'}(\Omega, w^*)$ is studied in[1] (respectively,[2]). Our aim in this paper is to prove the existence results of the problem $(\mathcal{P})$ by using the approach based on the strong convergence of the positive part u_ε^+ (resp. u_ε^-) of u_ε, where u_ε is a solution of the approximate problem. Note that, the same result is proved in[3] by using another approach based on the strong convergence of the truncations. On the other hand, in the non weighted case Boccardo Gallouët[5] have proved the existence of at least one solution for the problem $(\mathcal{P})$. Let us point out that another work in this direction can be found in.[4,6] Our results theorem generalizes those obtained in[5] and,[6] in the weighted case.
The present paper is organized as follows: In section 2, we give some preliminaries and some technical lemmas. In section 3, we give and prove our main general result. The fourth part is devoted to an example which illustrates our abstract hypothesis.

2. Preliminaries

Let Ω be a bounded open subset of $I\!R^N$ $(N \geq 1)$, let $1 < p < \infty$, and let $w = \{w_i(x)\}$, $0 \leq i \leq N$ be a vector of weight functions; i.e. every component $w_i(x)$ is a measurable function which is strictly positive a.e. in Ω. Further, we suppose in all our considerations that for $0 \leq i \leq N$,

$$w_i \in L^1_{\text{loc}}(\Omega) \tag{1}$$

$$w_i^{-\frac{1}{p-1}} \in L^1_{\text{loc}}(\Omega)\,. \tag{2}$$

We define the weighted space with weight γ on Ω as

$$L^p(\Omega, \gamma) = \{u = u(x),\ u\gamma^{1/p} \in L^p(\Omega)\}\,.$$

In this space, we define the norm

$$\|u\|_{p,\gamma} = \Big(\int_\Omega |u(x)|^p \gamma(x)\, dx\Big)^{1/p}.$$

We denote by $W^{1,p}(\Omega, w)$ the space of all real-valued functions $u \in L^p(\Omega, w_0)$ such that the derivatives in the sense of distributions satisfy

$$\frac{\partial u}{\partial x_i} \in L^p(\Omega, w_i) \text{ for all } i = 1, \ldots, N\,.$$

This set of functions forms a Banach space under the norm

$$\|u\|_{1,p,w} = \Big(\int_\Omega |u(x)|^p w_0(x)\, dx + \sum_{i=1}^N \int_\Omega |\frac{\partial u(x)}{\partial x_i}|^p w_i(x)\, dx \Big)^{1/p}. \quad (3)$$

To deal with the Dirichlet problem, we use the space

$$X = W_0^{1,p}(\Omega, w)$$

defined as the closure of $C_0^\infty(\Omega)$ with respect to the norm (3). Note that, $C_0^\infty(\Omega)$ is dense in $W_0^{1,p}(\Omega, w)$ and $(X, \|.\|_{1,p,w})$ is a reflexive Banach space. We recall that the dual space of the weighted Sobolev spaces $W_0^{1,p}(\Omega, w)$ is equivalent to $W^{-1,p'}(\Omega, w^*)$, where $w^* = \{w_i^* = w_i^{1-p'}\}$ $i = 0, \ldots, N$, and p' is the conjugate of p i.e. $p' = \frac{p}{p-1}$. For more details, we refer the reader to.[8]

Now, we state the following assumption.

(H1) The expression

$$|||u|||_X = (\sum_{i=1}^N \int_\Omega |\frac{\partial u(x)}{\partial x_i}|^p w_i(x)\, dx)^{1/p} \quad (4)$$

is a norm defined on X and is equivalent to the norm (3).

Note that $(X, |||.|||_X)$ is a uniformly convex (and thus reflexive) Banach space. There exist a weight function σ on Ω and a parameter q, $1 < q < \infty$, such that

$$\sigma^{1-q'} \in L^1_{\text{loc}}(\Omega), \quad (5)$$

with $q' = \frac{q}{q-1}$ and such that the Hardy inequality,

$$\Big(\int_\Omega |u(x)|^q \sigma\, dx \Big)^{1/q} \leq c \Big(\sum_{i=1}^N \int_\Omega |\frac{\partial u(x)}{\partial x_i}|^p w_i(x)\, dx \Big)^{1/p}, \quad (6)$$

holds for every $u \in X$ with a constant $c > 0$ independent of u. Moreover, the imbedding

$$X \hookrightarrow L^q(\Omega, \sigma), \quad (7)$$

determined by the inequality (6) is compact.

Remark 2.1. If we assume that $w_0(x) \equiv 1$ and that there exists $\nu \in]\frac{N}{p}, \infty[\cap [\frac{1}{p-1}, \infty[$ such that $w_i^{-\nu} \in L^1(\Omega)$ for all $i = 1, \ldots, N$, (which is an integrability condition, stronger than (2)), then

$$|||u|||_X = \Big(\sum_{i=1}^{N} \int_\Omega |\frac{\partial u(x)}{\partial x_i}|^p w_i(x)\, dx \Big)^{1/p}$$

is a norm defined on $W_0^{1,p}(\Omega, w)$ and equivalent to (3). Also we have that

$$W_0^{1,p}(\Omega, w) \hookrightarrow L^q(\Omega)$$

for $1 \le q < p_1^*$, $p\nu < N(\nu+1)$, and $q \ge 1$ is arbitrary for $p\nu \ge N(\nu+1)$ where $p_1 = \frac{p\nu}{\nu+1}$. Where $p_1^* = \frac{Np_1}{N-p_1} = \frac{Np\nu}{N(\nu+1)-p\nu}$ is the Sobolev conjugate of p_1 (see[8]). Thus the hypotheses (H1) is verified (for $\sigma \equiv 1$).

Now, we recall the following technical lemmas which are needed later. Note that this lemmas are proved in.[1]

Lemma 2.1. [1] *Let $g \in L^r(\Omega, \gamma)$ and let $g_n \in L^r(\Omega, \gamma)$, with $\|g_n\|_{r,\gamma} \le c$, $1 < r < \infty$. If $g_n(x) \to g(x)$ a.e. in Ω, then $g_n \rightharpoonup g$ in $L^r(\Omega, \gamma)$, where $\rightharpoonup$ denotes weak convergence and γ is a weight function on Ω.*

Lemma 2.2. [1] *Assume that (H1) holds. Let $F : \mathbb{R} \to \mathbb{R}$ be uniformly Lipschitzian, with $F(0) = 0$. Let $u \in W_0^{1,p}(\Omega, w)$. Then $F(u) \in W_0^{1,p}(\Omega, w)$. Moreover, if the set D of discontinuity points of F' is finite, then*

$$\frac{\partial(F \circ u)}{\partial x_i} = \begin{cases} F'(u)\frac{\partial u}{\partial x_i} & a.e. \text{ in } \{x \in \Omega : u(x) \notin D\} \\ 0 & a.e. \text{ in } \{x \in \Omega : u(x) \in D\}. \end{cases}$$

Lemma 2.3. [1] *Assume that (H1) holds. Let $u \in W_0^{1,p}(\Omega, w)$, and let $T_k(u)$, $k \in \mathbb{R}^+$, be the usual truncation then $T_k(u) \in W_0^{1,p}(\Omega, w)$. Moreover, we have*

$$T_k(u) \to u \text{ strongly in } W_0^{1,p}(\Omega, w).$$

Lemma 2.4. [1] *Assume that (H1) holds. Let $u \in W_0^{1,p}(\Omega, w)$, then $u^+ = \max(u, 0)$ and $u^- = \max(-u, 0)$ lie in $W_0^{1,p}(\Omega, w)$. Moreover, we have*

$$\frac{\partial(u^+)}{\partial x_i} = \begin{cases} \frac{\partial u}{\partial x_i}, & \text{if } u > 0 \\ 0, & \text{if } u \le 0 \end{cases}$$

$$\frac{\partial(u^-)}{\partial x_i} = \begin{cases} 0, & \text{if } u \ge 0 \\ -\frac{\partial u}{\partial x_i}, & \text{if } u < 0. \end{cases}$$

Lemma 2.5. [1] *Assume that (H1) holds. Let* (u_n) *be a sequence of* $W_0^{1,p}(\Omega, w)$ *such that* $u_n \rightharpoonup u$ *weakly in* $W_0^{1,p}(\Omega, w)$*. Then,* $u_n^+ \rightharpoonup u^+$ *weakly in* $W_0^{1,p}(\Omega, w)$ *and* $u_n^- \rightharpoonup u^-$ *weakly in* $W_0^{1,p}(\Omega, w)$*.*

3. Main result

Let A be the nonlinear operator from $W_0^{1,p}(\Omega, w)$ into the dual $W^{-1,p'}(\Omega, w^*)$ defined as

$$Au = -\operatorname{div}(a(x, u, \nabla u)),$$

where $a : \Omega \times I\!R \times I\!R^N \to I\!R^N$ is a Carathéodory vector-function satisfying the following assumptions:

(H2) For $i = 1, \dots, N$,

$$|a_i(x, s, \xi)| \leq \beta w_i^{1/p}(x)[k(x) + \sigma^{1/p'}|s|^{\frac{q}{p'}} + \sum_{j=1}^{N} w_j^{1/p'}(x)|\xi_j|^{p-1}] \quad (8)$$

$$[a(x, s, \xi) - a(x, s, \eta)](\xi - \eta) > 0 \text{ for all } \xi \neq \eta \in I\!R^N \quad (9)$$

$$a(x, s, \xi).\xi \geq \alpha \sum_{i=1}^{N} w_i|\xi_i|^p, \quad (10)$$

where $k(x)$ is a positive function in $L^{p'}(\Omega)$ and α, β are positive constants.

(H3) $g(x, s, \xi)$ is a Carathéodory function satisfying

$$g(x, s, \xi)s \geq 0, \quad (11)$$

$$|g(x, s, \xi)| \leq b(|s|)(\sum_{i=1}^{N} w_i|\xi_i|^p + c(x)), \quad (12)$$

There exists $\rho_1 > 0$ and $\rho_2 > 0$ such that, for $|s| \geq \rho_1$, $|g(x, s, \xi)| \geq \rho_2 \sum_{i=1}^{N} w_i|\xi_i|^p$. (13)

where $b : I\!R^+ \to I\!R^+$ is a continuous increasing function and $c(x)$ is positive function which lies in $L^1(\Omega)$.

Finally, we assume that

$$f \in L^1(\Omega). \quad (14)$$

Consider the nonlinear problem with Dirichlet boundary conditions

$$(\mathcal{P}) \begin{cases} Au + g(x, u, \nabla u) = f & \text{in } \mathcal{D}'(\Omega), \\ u \in W_0^{1,p}(\Omega, w), \quad g(x, u, \nabla u) \in L^1(\Omega). \end{cases}$$

We state our main result as follows.

Theorem 3.1. *Under assumptions (H1)-(H3) and (14), there exists at least one solution of the problem (P).*

Remark 3.1. The statement of Theorem 3.1 generalizes in the weighted case the one in.[5,6]

Remark 3.2. In the case $f \in W^{-1,p'}(\Omega, w)$, it is true that $g(x, u, \nabla u)u \in L^1(\Omega)$, which in contrast is in general false if we assume that $f \in L^1(\Omega)$ (cf. Remark 3[5]).

Remark 3.3. In the case $f \geq 0$, the solution of the problem $(\mathcal{P})$ is positive it sufficient to take $v = -T_k(u^-)$ $k > 0$.

In order to prove Theorem 3.1, we needed the following lemma. Which is proved in[1]

Lemma 3.1. [1] *Assume that (H1) and (H2) are satisfied, and let (u_n) be a sequence in $W_0^{1,p}(\Omega, w)$ such that $u_n \rightharpoonup u$ weakly in $W_0^{1,p}(\Omega, w)$ and*

$$\int_\Omega [a(x, u_n, \nabla u_n) - a(x, u_n, \nabla u)]\nabla(u_n - u)\, dx \to 0. \tag{15}$$

Then, $u_n \to u$ in $W_0^{1,p}(\Omega, w)$.

Proof of Theorem 3.1

Step (1) The approximate problem and a priori estimate

Let f_ε be a sequence of smooth functions which strongly converges to f in $L^1(\Omega)$ and $\|f_\varepsilon\|_{L^1(\Omega)} \leq c_1$ for some constants c_1.

Now, consider the following approximate problem

$$(\mathcal{P}_\varepsilon) \begin{cases} A(u_\varepsilon) + g(x, u_\varepsilon, \nabla u_\varepsilon) = f_\varepsilon \\ u_\varepsilon \in W_0^{1,p}(\Omega, w). \end{cases}$$

Which has a solution by the Theorem 3.1 in.[1] Multiplying $(\mathcal{P}_\varepsilon)$ by $T_k(u_\varepsilon) \in X$ and since $\int_\Omega g(x, u_\varepsilon, \nabla u_\varepsilon)T_k(u_\varepsilon) \geq 0$, we obtain

$$\int_\Omega a(x, u_\varepsilon, \nabla T_k(u_\varepsilon))\nabla T_k(u_\varepsilon) \leq \int_\Omega f_\varepsilon T_k(u_\varepsilon) \leq kc_1.$$

In view of (10), we have

$$\sum_{i=1}^N \int_\Omega w_i |\frac{\partial T_k(u_\varepsilon)}{\partial x_i}|^p \, dx \leq \frac{k}{\alpha} c_1$$

i.e.,

$$|||T_k(u_\varepsilon)|||^p \leq c_2. \tag{16}$$

On the other hand, we have,

$$k\int_{\{|u_\varepsilon|>k\}} |g(x,u_\varepsilon,\nabla u_\varepsilon)| \leq \int_\Omega |f_\varepsilon||T_k(u_\varepsilon)|\,dx \leq kc_1, \tag{17}$$

which implies

$$\int_{\{|u_\varepsilon|>k\}} \sum_{j=1}^N |\frac{\partial u_\varepsilon}{\partial x_i}|w_i \leq \frac{1}{\rho_2}\int_{\{|u_\varepsilon|>k\}} |f_\varepsilon|\,dx. \tag{18}$$

Then, by (13), (16) and (17) with $k > \rho_1$, we obtain

$$\begin{aligned} |||u_\varepsilon|||_X^p &= \sum_{i=1}^N \int_{\{|u_\varepsilon>k|\}} w_i|\frac{\partial u_\varepsilon}{\partial x_i}|^p\,dx + \sum_{i=1}^N \int_\Omega w_i|\frac{\partial T_k(u_\varepsilon)}{\partial x_i}|^p\,dx \\ &\leq \frac{1}{\rho_2}\int_{\{|u_\varepsilon|>k\}} |g(x,u_\varepsilon,\nabla u_\varepsilon)| + c_3 + c_2 \leq c_4, \end{aligned}$$

where $c_i \;\; i = 1, 2, \ldots,$ are various positive constants. Then

$$|||u_\varepsilon|||_X \leq c$$

where c is some positive constant. Hence, we can extract a subsequence still denoted by u_ε such that,

$$u_\varepsilon \rightharpoonup u \text{ in } W_0^{1,p}(\Omega, w) \text{ and a.e. in } \Omega. \tag{19}$$

Step (2) Convergence of the positive part of u_ε.
Our objective in this step is to prove that

$$u_\varepsilon^+ \to u^+ \text{ strongly in } W_0^{1,p}(\Omega, w). \tag{20}$$

Assertion (i) We prove that:

$$\lim_{\varepsilon\to 0}\int_\Omega [a(x,u_\varepsilon,\nabla u_\varepsilon^+) - a(x,u_\varepsilon,\nabla u^+)]\nabla(u_\varepsilon^+ - u^+)^+\,dx = 0. \tag{21}$$

Let k be a positive constant greater than ρ_1. We use $v_\varepsilon = T_k(u_\varepsilon^+ - u^+)^+$ as a test function in $(\mathcal{P}_\varepsilon)$, this is an admissible test function which due by Lemmas 2.3 and 2.4. Multiplying $(\mathcal{P}_\varepsilon)$ by $v_\varepsilon = T_k(u_\varepsilon^+ - u^+)^+$, we obtain

$$\langle Au_\varepsilon, v_\varepsilon\rangle + \int_\Omega g(x,u_\varepsilon,\nabla u_\varepsilon)v_\varepsilon\,dx = \langle f_\varepsilon, v_\varepsilon\rangle.$$

If $v_\varepsilon(x) > 0$, we have $u_\varepsilon(x) > 0$ and from (11) $g(x,u_\varepsilon,\nabla u_\varepsilon) \geq 0$, then $\langle Au_\varepsilon, v_\varepsilon\rangle \leq \langle f_\varepsilon, v_\varepsilon\rangle$ i.e.,

$$\int_\Omega a(x,u_\varepsilon,\nabla u_\varepsilon)\nabla v_\varepsilon\,dx \leq \langle f_\varepsilon, v_\varepsilon\rangle.$$

Since $u_\varepsilon = u_\varepsilon^+$ in $\{x \in \Omega, u_\varepsilon^+(x) > u^+(x)\}$, then

$$\int_\Omega a(x, u_\varepsilon, \nabla u_\varepsilon^+) \nabla T_k(u_\varepsilon^+ - u^+)\, dx \le \langle f_\varepsilon, T_k(u_\varepsilon^+ - u^+)\rangle,$$

which implies

$$\lim_{\varepsilon\to 0} \int_\Omega [a(x, u_\varepsilon, \nabla u_\varepsilon^+) - a(x, u_\varepsilon, \nabla u^+)] \nabla T_k(u_\varepsilon^+ - u^+)^+\, dx = 0. \tag{22}$$

On the other hand, since again that $u_\varepsilon^+(x) = u_\varepsilon(x)$ in the set $\{x \in \Omega, u_\varepsilon^+(x) - u^+(x) > 0\}$, then by using (8) and Young's inequality, we obtain

$$\begin{aligned} &\int_{u_\varepsilon^+ - u^+ > k} [a(x, u_\varepsilon, \nabla u_\varepsilon^+) - a(x, u_\varepsilon, \nabla u^+)] \nabla (u_\varepsilon^+ - u^+)^+\, dx \\ &\qquad \le \int_{u_\varepsilon > k} [a(x, u_\varepsilon, \nabla u_\varepsilon^+) - a(x, u_\varepsilon, \nabla u^+)] \nabla (u_\varepsilon^+ - u^+)^+\, dx \\ &\qquad \le c\Big(\int_{u_\varepsilon > k} k^{p'}(x)\, dx + \int_{u_\varepsilon > k} |u_\varepsilon|^q \sigma\, dx \\ &\quad + \textstyle\int_{u_\varepsilon > k} \sum_{i=1}^N |\frac{\partial u_\varepsilon}{\partial x_i}|^p w_i\, dx + \int_{u_\varepsilon > k} \sum_{i=1}^N |\frac{\partial u^+}{\partial x_i}|^p w_i\, dx \Big) \end{aligned}$$

thanks to (18), we deduce

$$\begin{aligned} &\int_{u_\varepsilon^+ - u^+ > k} [a(x, u_\varepsilon, \nabla u_\varepsilon^+) - a(x, u_\varepsilon, \nabla u^+)] \nabla (u_\varepsilon^+ - u^+)^+\, dx \\ &\qquad \le c\Big(\textstyle\int_{u_\varepsilon > k} k^{p'}(x)\, dx + \int_{u_\varepsilon > k} |u_\varepsilon|^q \sigma\, dx + \int_{u_\varepsilon > k} |f_\varepsilon|\, dx \\ &\quad + \textstyle\int_{u_\varepsilon > k} \sum_{i=1}^N |\frac{\partial u^+}{\partial x_i}|^p w_i\, dx \Big). \end{aligned} \tag{23}$$

If k tends to infinity the right hand side of (23) converges to zero with respect to ε.
Combining (22) and (23), we conclude (21).
Assertion (ii) We claim that

$$\lim_{\varepsilon\to 0} \int_\Omega [a(x, u_\varepsilon, \nabla u_\varepsilon^+) - a(x, u_\varepsilon, \nabla u^+)] \nabla (u_\varepsilon^+ - u^+)^-\, dx = 0. \tag{24}$$

Indeed, take $z_\varepsilon = u_\varepsilon^+ - T_k(u^+)$, we shall use the test function $v_\varepsilon = \varphi_\lambda(z_\varepsilon^-)$ with $\varphi_\lambda(s) = se^{\lambda s^2}$ in $(\mathcal{P}_\varepsilon)$. If $v_\varepsilon(x) \neq 0$, we have $0 \le u_\varepsilon^+(x) \le k$, hence $z_\varepsilon^- \in L^\infty(\Omega)$ and since $z_\varepsilon^- \in W_0^{1,p}(\Omega, w)$, hence by Lemma 2.2, we have $v_\varepsilon \in W_0^{1,p}(\Omega, w)$. Multiplying $(\mathcal{P}_\varepsilon)$ by v_ε, we obtain

$$\int_\Omega a(x, u_\varepsilon, \nabla u_\varepsilon) \nabla z_\varepsilon^- \varphi_\lambda'(z_\varepsilon^-)\, dx + \int_\Omega g(x, u_\varepsilon, \nabla u_\varepsilon) \varphi_\lambda(z_\varepsilon^-)\, dx = \langle f_\varepsilon, \varphi_\lambda(z_\varepsilon^-)\rangle. \tag{25}$$

Firstly, we study the term of the right hand side, since $\varphi_\lambda(z_\varepsilon^-) \neq 0$ where $0 \leq u_\varepsilon^+(x) \leq k$, then $\varphi_\lambda(z_\varepsilon^-)$ is bounded in $L^\infty(\Omega)$, then by Lebesgue dominated convergence theorem, we obtain

$$\lim_{\varepsilon\to 0}\int_\Omega f_\varepsilon \varphi_\lambda(z_\varepsilon^-)\,dx = \int_\Omega f\varphi_\lambda(u^+ - T_k(u^+))^-)\,dx.$$

On the other hand, we study the term of left hand side as in the proof of assertion (4.6) in.[1] Consequently, we conclude that

$$-\limsup_{\varepsilon\to 0}\int_\Omega [a(x,u_\varepsilon,\nabla u_\varepsilon^+) - a(x,u_\varepsilon,\nabla T_k(u)^+)]\nabla(u_\varepsilon^+ - T_k(u)^+)^- \leq 0. \quad (26)$$

Finally, we deduce the assertion (24) as in the proof of (21) in.[5]
Combining (21) and (24), we obtain

$$\lim_{\varepsilon\to 0}\int_\Omega [a(x,u_\varepsilon,\nabla u_\varepsilon^+) - a(x,u_\varepsilon,\nabla u^+)]\nabla(u_\varepsilon^+ - u^+)\,dx = 0. \quad (27)$$

Which and using Lemma 3.1, we conclude (20).

Step(3) Convergence of the negative part of u_ε.

The proof of this step is similar to the step (2), it is sufficient to take $v_\varepsilon = T_k(u_\varepsilon^- - u^-)^+$ and $v_\varepsilon = \varphi_\lambda(u_\varepsilon^- - u^-)^-$ as test function in $(\mathcal{P}_\varepsilon)$. Then, we can prove that

$$u_\varepsilon^- \to u^- \text{ strongly in } W_0^{1,p}(\Omega, w). \quad (28)$$

Step(4) Strong convergence of u_ε and passing to the limit.

From (20) and (28), we conclude that for a subsequence,

$$u_\varepsilon \to u \text{ strongly in } W_0^{1,p}(\Omega, w) \text{ and } a.e. \text{ in } \Omega. \quad (29)$$

Consequently, we deduce that

$$\nabla u_\varepsilon \to \nabla u \quad \text{in } \Pi_{i=1}^N L^p(\Omega, w_i) \text{ and a.e. in } \Omega.$$

Reasoning as in the proof of Theorem 3.2 in,[3] we can prove that

$$g(x,u_\varepsilon,\nabla u_\varepsilon) \to g(x,u,\nabla u) \text{ strongly in } L^1(\Omega). \quad (30)$$

Finally, thanks to (29) and (30), it is easy to pass to the limit in

$$\int_\Omega a(x,u_\varepsilon,\nabla u_\varepsilon)\nabla v\,dx + \int_\Omega g(x,u_\varepsilon,\nabla u_\varepsilon)v\,dx = \int_\Omega f_\varepsilon v\,dx$$

for all $v \in W_0^{1,p}(\Omega, w) \cap L^\infty(\Omega)$.
Then, we obtain

$$\int_\Omega a(x,u,\nabla u)\nabla v\,dx + \int_\Omega g(x,u,\nabla u)v\,dx = \int_\Omega fv\,dx$$

for all $v \in W_0^{1,p}(\Omega, w) \cap L^\infty(\Omega)$.

3.1. *Example*

Some ideas of this example come from.[7] Let Ω be a bounded domain of $I\!R^N$ $(N \geq 1)$, satisfying the cone condition. Let us consider the Carathéodory functions:

$$a_i(x,s,\xi) = w_i|\xi_i|^{p-1}sgn(\xi_i) \qquad \text{for } i = 1,...,N$$

$$g(x,s,\xi) = \rho s|s|^r \sum_{i=1}^{N} w_i|\xi_i|^p, \qquad \rho > 0,\ r > 0$$

where $w_i(x)$ $(i = 0,1,...,N)$ are a given weight functions strictly positive almost everywhere in Ω. We shall assume that the weight functions satisfy, $w_i(x) = w(x), \quad x \in \Omega, \quad$ for all $i = 0,...,N$. Then, we can consider the Hardy inequality (6) in the form,

$$\left(\int_\Omega |u(x)|^q \sigma(x)\ dx\right)^{\frac{1}{q}} \leq c\left(\int_\Omega |\nabla u(x)|^p w\right)^{\frac{1}{p}}.$$

It is easy to show that the $a_i(x,s,\xi)$ are Carathéodory functions satisfying the growth condition (8) and the coercivity (10). Also the Carathéodory function $g(x,s,\xi)$ satisfies the conditions (11), (12) and (13) with $|s| \geq \rho_1 = 1$ and $\rho_2 = \rho > 0$. On the other hand, the monotonicity condition is verified, in fact,

$$\sum_{i=1}^{N}\left(a_i(x,s,\xi) - a_i(x,s,\hat{\xi})\right)\left(\xi_i - \hat{\xi}_i\right)$$

$$= w(x)\textstyle\sum_{i=1}^{N}\left(|\xi_i|^{p-1}sgn\xi_i - |\hat{\xi}_i|^{p-1}sgn\hat{\xi}_i\right)\left(\xi_i - \hat{\xi}_i\right) > 0$$

for almost all $x \in \Omega$ and for all $\xi, \hat{\xi} \in I\!R^N$ with $\xi \neq \hat{\xi}$, since $w > 0$ *a.e.* in Ω. In particular, let us use the special weight functions w and σ expressed in terms of the distance to the boundary $\partial\Omega$. Denote $d(x) = dist(x, \partial\Omega)$ and set

$$w(x) = d^\lambda(x),\ \sigma(x) = d^\mu(x).$$

In this case, the Hardy inequality reads

$$\left(\int_\Omega |u(x)|^q\ d^\mu(x)\ dx\right)^{\frac{1}{q}} \leq c\left(\int_\Omega |\nabla u(x)|^p\ d^\lambda(x)\ dx\right)^{\frac{1}{p}}.$$

The corresponding imbedding is compact if:
i) For, $1 < p \leq q < \infty$,

$$\lambda < p-1, \quad \frac{N}{q} - \frac{N}{p} + 1 \geq 0, \quad \frac{\mu}{q} - \frac{\lambda}{p} + \frac{N}{q} - \frac{N}{p} + 1 > 0. \tag{31}$$

ii) For, $1 \leq q < p < \infty$,

$$\lambda < p - 1, \quad \frac{\mu}{q} - \frac{\lambda}{p} + \frac{1}{q} - \frac{1}{p} + 1 > 0. \tag{32}$$

Remark 3.4. Condition (31) or (32) are sufficient for the compact imbedding (7) to hold (see for example[7] Example 1,[8] Example 1.5, p.34 and[10] theorem 19.17 and 19.22).

Finally, the hypotheses of theorem 3.1 are satisfied, therefore the problem $(\mathcal{P})$ has at least one solution.

References

1. Y. Akdim, E. Azroul and A. Benkirane, *Existence of Solution for Quasilinear Degenerated Elliptic Equations,* Electronic J. Diff. Equ., Vol. **2001** N 71, (2001) pp 1-19.
2. Y. Akdim, E. Azroul and A. Benkirane, *Existence of Solution for Quasilinear Degenerated Elliptic Unilateral Problems,* Annale Mathématique Blaise Pascal Vol. 10 (2003) pp 1-20.
3. Y. Akdim, E. Azroul and A. Benkirane, *Existence results for Quasilinear Degenerated equation via strong convergence of trancations,* Revista Matematica Complutense , Vol.17 (2004), pp 359-379..
4. A. Bensoussan, L. Boccardo and F. Murat, *On a non linear partial differential equation having natural growth terms and unbounded solution,* Ann. Inst. Henri Poincaré **5** $N4$ (1988), 347-364.
5. L. Boccardo and T. Gallouët, *Strongly MonlinearElliptic Equations Having Natural Growth Terms and L^1 Data,*Nonlinear Analysis Theory Methods and Applications, Vol **19**, $N6$ (1992) 573-579.
6. L. Boccardo, T. Gallouët and F. Murat, *A unified presentation of two existence results for problems with natural growth,*in Progress in PDE, the Metz surveys 2, M. Chipot editor, Research in Mathematics, Longman, **296** (1993), 127-137.
7. P. Drabek, A. Kufner and V. Mustonen, *Pseudo-monotonicity and degenerated or singular elliptic operators,* Bull. Austral. Math. Soc. Vol. **58** (1998), 213-221.
8. P. Drabek, A. Kufner and F. Nicolosi, *Non linear elliptic equations, singular and degenerate cases,* University of West Bohemia, (1996).
9. P. Drabek and F. Nicolosi, *Existence of Bounded Solutions for Some Degenerated Quasilinear Elliptic Equations,* Annali di Mathematica pura ed applicata (IV), Vol. CLXV (1993), pp. 217-238.
10. B. Opic and A. Kufner, *Hardy-type inequalities, Pitman Research Notes in Mathematics Series **219**(Longman Scientific and Technical, Harlow, 1990).*

Combined fuzzy sliding mode controller for a class of nonlinear systems

Y. Alaoui Hafidi

Laboratoire d'Electronique Signaux Systèmes et Informatique (L.E.S.S.I)
Département de physique, Faculté des Sciences Dhar Mehraz
B.P: 1796, 30000 Fes-Atlas, Morocco

J. Boumhidi

Laboratoire d'Informatique, Mathématiques, Automatique et d'Optoélectronique (L.I.M.A.O)
Faculté Polydisciplinaire de Taza
Route d'Oujda, B.P: 1223 Taza Morocco
E-mail: jboumhidi@yahoo.fr

I. Boumhidi

Laboratoire d'Electronique Signaux Systèmes et Informatique (L.E.S.S.I)
Département de physique, Faculté des Sciences Dhar Mehraz
B.P: 1796, 30000 Fes-Atlas, Morocco

In this study, we present an adaptive combined fuzzy sliding mode controller for a class of nonlinear systems with unknown nonlinear dynamics. The proposed control strategy is based on the fuzzy logic and sliding mode control technique (SMC). The adaptive fuzzy logic system type Takagi-Sugeno (T-S) is used to approximate the unknown dynamics of system. In order to assure the robustness on stability and tracking performance, the sliding mode control term is added in the control law. The latter consists to suppress the influence of external disturbances and attenuate fuzzy approximation error. Simulation results illustrate the good performances of the system when the proposed control technique is applied.

Keywords: Fuzzy logic; Sliding mode control; Takagi-Sugeno-type system.

1. Introduction

Fuzzy logic, as one of the most useful approaches for utilizing expert knowledge, has been an active filed of research during the past decade,[16] Fuzzy logic control has found promising applications for a wide based on the universal approximation theorem,[5] stable variety of systems specifically applicable to plants that are mathematically poorly modeled.[7] Based on the universal approximation capability, many effective adaptive fuzzy control schemes have been developed to incorporate with human expert knowledge information in a systematic way, which can also guarantee stability and performance criteria.[9] The SMC method is proposed to provide robust controller systems,[2],[10][11] However, this approach needs a nominal mathematical model for the control law. Fuzzy control technique has been successfully applied to many industrial systems where no accurate mathematical models of the systems under control are available and human experts are available to provide linguistic fuzzy control rules or linguistic fuzzy descriptions about the systems,[7][9] The apparent similarities between sliding mode control and fuzzy control motivate considerable research efforts in combining the two approaches for achieving more superior performances such as overcoming some limitations of the traditional sliding mode control,[3][8] In this paper, adaptive fuzzy controller is proposed for a class of nonlinear systems with unknown nonlinear dynamics. The adaptive fuzzy model type T-S is used to approximate the unknown dynamics systems. The adjustable fuzzy parameters are updated on line by the adaptive algorithm. The stability and convergence analysis is ensured from the Lyapunov approach. The added term control based on SMC approach is designed to attenuate the influence of external disturbances and to remove the fuzzy approximation error. This paper is organized as follows: The problem formulation is presented in section 2. In section 3, the fuzzy SMC design is proposed for nonlinear systems with unknown dynamics. In Section 4, simulations example is proposed to shown the effectiveness of the proposed method.

2. Problem formulation

Consider a nonlinear system described by:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_3 \\ . \\ . \\ . \\ \dot{x}_n = f(\underline{x}, t) + bu + d(t) \\ y = x_1 \end{cases} \tag{1}$$

or equivalently

$$\begin{cases} x^{(n)} = f(\underline{x}, t) + bu + d(t) \\ y = x \end{cases} \quad \text{with} \quad x^{(n)} = \frac{d^n}{dt^n} \tag{2}$$

$\underline{x} = [x, \dot{x}, \ldots, x^{(n-1)}]^T = [x_1, x_2, \ldots, x_n]^T \in I\!R^n$ is the state vector of the systems which is assumed to be available for measurements, $u \in I\!R$ and $y \in I\!R$ are respectively the input and the output of the system. $f(\underline{x}, t)$ is unknown and nonlinear function, and b is positive constant. $d(t)$ is unknown external disturbances. The control problem is to obtain the state x for tracking a desired y_r. Define the tracking error $e = y - y_r$. It is desired that the output error of the system follow: $e^{(n)} + k_{n-1}e^{(n-1)} + \cdots + k_0 e = 0$ to ensure a good tracking. With k_i , $i = 1, 2, \ldots, n-1$ are selected so that the associated characteristic equation has roots in the open left-half complex. When the system (1) is well known without disturbances and $b \neq 0$, the control law is designed to guarantee that $\lim_{t\to\infty} e = 0$ as follows:

$$u = \frac{1}{b}(-f(x, t) + y_r^{(n)} - \sum\nolimits_{i=1}^{n-1} k_i e^{(i)}). \tag{3}$$

However, in practice, this objective can not be achieved the dynamic of system $f(\underline{x}, t)$ is unknown and the design of the control law (3) is confronted with many problems due to environment changes, modelling errors and unmodelled dynamics. To solve this problem, we propose a fuzzy system to approximate the unknown dynamic and using the SMC technique and Lyapunov approach to guarantee the stability and the tracking performance.

3. Fuzzy sliding mode control design

In this section, the adaptive fuzzy control is designed for a class of nonlinear system with unknown nonlinear dynamics and disturbances. The strategy of control is based on fuzzy logic and SMC approach to ensure stability and good tracking. The dynamics system $f(\underline{x}, t)$ is approximated by the adaptive fuzzy model type T-S. The fuzzy parameters can be tuned on line by adaptive law based on Lyapunov approach. To ensure stability and tracking performance, the auxiliary control action is incorporated in the control law (3) to attenuate the external disturbances and remove fuzzy approximation error. The design of this compensation control is derived from SMC and the Lyapunov approach. The basic configuration of a fuzzy logic system consists of a fuzzifier, a fuzzy rule base, a fuzzy inference engine and a defuzzifier. The fuzzy inference engine uses the fuzzy IF-THEN rules to perform a mapping from an input vector $\underline{x} = [x_1, x_2, \ldots, x_n]^T$ to an output scalar $\hat{f}$. The fuzzy rule base consists of a collection of fuzzy IF-THEN rules in the following form:

$$R^j : \; if\, x_1 \; is \; F_1^j \, and \ldots \, and\, x_n \; is \; F_n^j, \; then \; \hat{f} \; is \; \theta_j \; j = 1, \ldots, M \tag{4}$$

where F_i^j, $i = 1, \ldots, n$ are fuzzy variables characterized by membership functions $\mu_{F_i^j}(x_i)$ and θ_i is the corresponding value of the output fuzzy singleton.

The output of the fuzzy system with singleton fuzzification, product inference and center average defuzzification can be expressed as:[7]

$$\hat{f}(\underline{x}, \underline{\theta}) = \frac{\sum_{j=1}^{M} \underline{\theta}_j \left(\prod_{i=1}^{n} \mu_{F_i^j}(x_i)\right)}{\sum_{j=1}^{M} \left(\prod_{i=1}^{n} \mu_{F_i^j}(x_i)\right)} = \underline{\theta}^T \xi(\underline{x}) \tag{5}$$

M is the total number of the fuzzy rules, $\underline{\theta} = [\theta_1, \theta_2, \ldots, \theta_M]^T$ is adjustable parameter vector,$\xi(\underline{x}) = [\xi_1(x), \ \xi_2(x), \ldots, \xi_M(x)]^T$ is the vector of the fuzzy basis functions [7]:

$$\xi_j(x) = \left(\prod_{i=1}^{n} \mu_{F_i^j}(x_i)\right) \Big/ \sum_{j=1}^{M} \left(\prod_{i=1}^{n} \mu_{F_i^j}(x_i)\right) \tag{6}$$

Define the optimal parameters vectors and fuzzy approximation error as:

$$\underline{\theta}^* = \arg \min_{\theta \in \Omega_f} \left(\sup_{x \in I\!R^n} \left| f(\underline{x}, t) - \hat{f}(\underline{x}, \underline{\theta}) \right| \right) \tag{7}$$

$\Omega_f = \{\underline{\theta} \in I\!R^n \ \ \|\underline{\theta}\| \leq M_f\}$ is the convex compact sets which contain feasible parameter sets for $\underline{\theta}$, with M_f is given constants. Define a time varying sliding surface $s(\underline{x}, t) = 0$, in the state space $I\!R^n$ by the scalar equation:

$$s(\underline{x}, t) = e_0^{(n-1)} + k_{n-1} e_0^{(n-2)} + \cdots + k_1 e_0 \tag{8}$$

Assumptions:

i) There exist a positive constant $f_{\max}$ such that $\left| f(\underline{x}, t) - \hat{f}(\underline{x}, \underline{\theta}) \right| < f_{\max}$.
ii) The positive constant k_s is chosen so that the following condition: $k_s \geq f_{\max} + d_{\max}$ is satisfied with $d_{\max}$ is a positive upper bound of the external disturbances.

The proposed control law is given by:

$$u = \hat{u}(\underline{x}, \underline{\theta}) + u_s \tag{9}$$

The primary control is represented as follows:

$$\hat{u}(\underline{x}, \underline{\theta}) = \frac{1}{b}\left(-\hat{f}(\underline{x}, \underline{\theta}) + y_r^{(n)} - \sum_{i=1}^{n-1} k_i e^{(i)}\right) \tag{10}$$

The auxiliary control is given as:

$$u_s = -k_s \tanh\left(\frac{s}{\varepsilon}\right) \tag{11}$$

with $\tanh\left(\frac{s}{\varepsilon}\right)$ is the hyperbolic tangent of $\frac{s}{\varepsilon}$ and ε is a small positive constant. The adjustable fuzzy parameters of $\hat{f}(\underline{x}, \underline{\theta})$ are tuned on line using the Lyapunov approach. In order to guarantee that the parameters are bounded, we introduce the projection algorithm [9] to restrict them in the closed set Ω .

$$\dot{\theta} = \begin{cases} -\gamma s\xi(x) \text{ if } (|\theta| < M \text{ or } (|\theta| = M \text{ and } \gamma s\xi^T\xi(x) > 0)) \\ -\gamma s\xi(x) + \gamma s\dfrac{\theta\theta^T\xi(x)}{|\theta|^2} \end{cases} \tag{12}$$

γ is a positive constant.

Theorem 3.1. *Consider the nonlinear system (1), satisfying the assumptions i) and ii). The fuzzy SMC law is chosen as (9) with The closed-loop system is stable in sense that all the signals are bounded and the tracking performance is achieved.*

Proof:

Consider the following Lyapunov function:

$$V = \frac{1}{2}s^2 + \frac{1}{2\gamma}\underline{\Phi}^T\underline{\Phi} \tag{13}$$

With $\Phi = \underline{\theta} - \underline{\theta}^*$. The time derivative of V equals:

$$\begin{aligned} \dot{V} &= s\dot{s} + \frac{1}{\gamma}\underline{\Phi}^T\underline{\dot{\Phi}} \\ &= s((f(\underline{x}, t) - \hat{f}(\underline{\theta}, \underline{x})) + d(t) + u_s) + \frac{1}{\gamma}\underline{\Phi}^T\underline{\dot{\Phi}} \\ &= s((f(\underline{x}, t) - \hat{f}^*(\underline{\theta x})) + (\hat{f}^*(\underline{\theta}, \underline{x}) - f(\underline{\theta}, \underline{x})) + d(t) + u_s) + \frac{1}{\gamma}\underline{\Phi}^T\underline{\dot{\Phi}} \\ &= s(\underline{\Phi}^T\xi(\underline{x}) + (\hat{f}^*(\underline{\theta}, \underline{x}) - f(\underline{x}, t)) + d(t) + u_s) + \frac{1}{\gamma}\underline{\Phi}^T\underline{\dot{\Phi}} \\ &= \tfrac{1}{\gamma}\underline{\Phi}^T(\underline{\dot{\Phi}} + \gamma\, s\xi(\underline{x})) + s((\hat{f}^*(\underline{\theta}, \underline{x}) - f(\underline{x}, t)) + d(t) + u_s) \\ \dot{V} &\leq \tfrac{1}{\gamma}\underline{\Phi}^T(\underline{\dot{\Phi}} + \gamma\, s\xi(\underline{x})) + |s|\,(f_{\max} + d_{\max}) + s\, u_s \end{aligned} \tag{14}$$

where $\dot{\Phi} = \dot{\theta}$; $\dot{V} \leq \frac{1}{\gamma}\underline{\Phi}^T(\underline{\dot{\theta}} + \gamma\, s\xi(\underline{x})) + |s|\,(f_{\max} + d_{\max}) + s\, u_s$.

The adjustable fuzzy parameters of $\hat{f}(\underline{x}, \underline{\theta})$ is chosen as follows: $\underline{\dot{\theta}} = -\gamma\, s\xi(\underline{x})$ Then we have:

$$\dot{V} \leq |s|\,(f_{\max} + d_{\max}) + s\, u_s \tag{15}$$

if $|s| > \varepsilon$, $\tanh\left(\frac{s}{\varepsilon}\right) = sgn(s)$, then $u_s = -k_s sgn(s)$ we obtain $\dot{V} \leq |s|\,(f_{\max} + d_{\max}) - k_s\,|s|$,

By choosing the positive constant $k_s \geq f_{\max} + d_{\max}$ we obtain $\dot{V} \leq 0$.

However, in a small $\varepsilon-$vicinity of the origin, the so called boundary layer ($|s| \leq \varepsilon$), is a smooth continuous function ($\tanh\left(\frac{s}{\varepsilon}\right) \neq \mathrm{sgn(s)}$). The system trajectories are confined to a boundary layer of the sliding manifold $s = 0$ [10].

4. Simulation results

In order to test the proposed controller algorithm, we consider the regulation problem of a nonlinear servomechanism,[4] which is described in space state as:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = f(\underline{x}, t) + u(t) + d(t) \\ y = x_1 \end{cases}$$

or equivalently

$$\begin{cases} x^{(2)} & = f(\underline{x}, t) + b\ u(t) + d(t) \\ y & = x \end{cases}$$

where $f(x,t) = -x_2 - 0.4\sin(x_1)$, $y_r = \frac{\pi}{3}\cos(0.025t)$ and $d(t) = 0.01\, rand\,(t_0, t_f)$ with $t_0 = 0\ and\ t_f = 100s$.
The control objective is to maintain the system to track the desired angle trajectory y_r.
The sliding surface is defined as: $\sigma(\underline{x}, t) = k_1 e_0(t) + k_2 \dot{e}_0(t)$ with $k_1 = 5$ and $k_2 = 1$.
In the first step, we need to define some fuzzy sets to cover the state space. The choice of the number of fuzzy set and the constant M is related to knowledge of expert on the system. For simplicity, we consider $M = 6$ and the fuzzy membership functions figure (1) are chosen as: Then there are 9 rules to approximate the primary control law . The fuzzy rules are defined by the following linguistics description:

$$R^l: \text{ If } x_1 \text{ is } F_1^l \text{ and } x_2\ is\ F_2^l \text{ then } \hat{f}^l \text{ is } \theta_l.$$

By using the singleton fuzzification, product inference and COG method defuzzification, the primary control is given by :

$$\hat{f}(x,\theta) = \frac{\sum\limits_{l=1}^{9} \theta_l \prod\limits_{i=1}^{2} \mu_{F_i^l}}{\sum\limits_{l=1}^{9} \prod\limits_{i=1}^{2} \mu_{F_i^l}} = \underline{\theta}^T \xi(\underline{x})$$

The auxiliary gain control is chosen as: $k_5 = 4$.

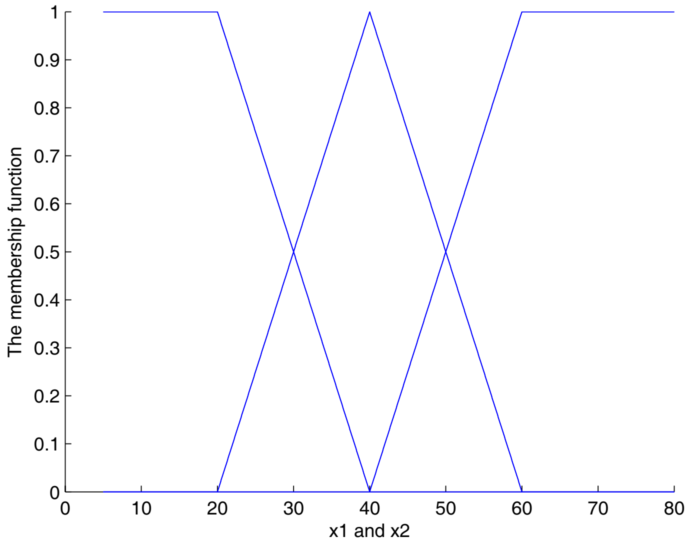


Fig. 1. Membership functions of x_1 and x_2.

It can be seen on figure 3 that a good tracking is obtained in presence of nonlinearity and disturbances. The corresponding fuzzy control signal is given on figure 2.

5. Conclusion

In this paper, fuzzy SMC algorithm has been proposed for a class of unknown nonlinear systems. We introduced the fuzzy system to approximate the dynamics systems. Moreover, the fuzzy parameters can be tuned online by the adaptive law based on Lyapunov synthesis. The added control term is incorporated in the control for ensuring stability, tracking in the presence of extern disturbances and removes the fuzzy approximation error. The simulation results shown that the proposed control methodology is effective.

References

1. J. S. Wang and C.S.G. Lee. "Sel-Adaptive Neuro-Fuzzy Inference Systems For Classification Application". IEEE Transactions on Fuzzy Systems, volum.10, no.6: pp.790-802, 2002.
2. C. Edwards and S.K. Spurgeon. "Sliding Mode Control-Theory and Application". London, U.K. Taylor Francis, 1998.

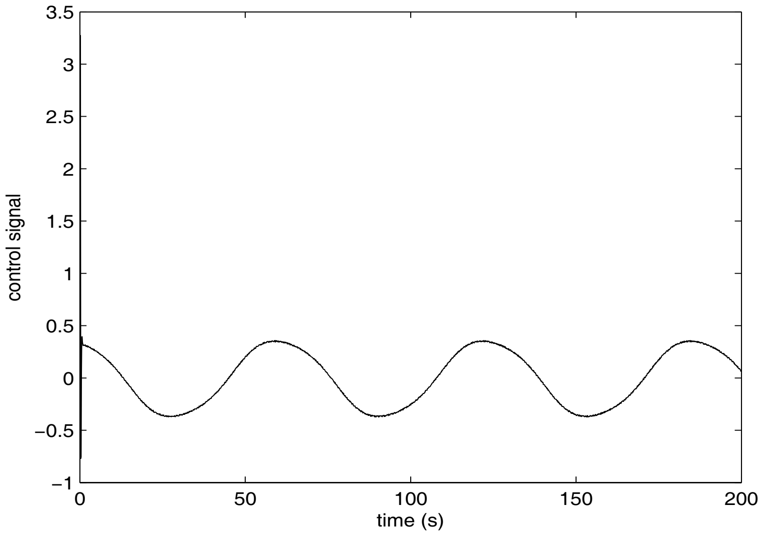


Fig. 2. The control signal.

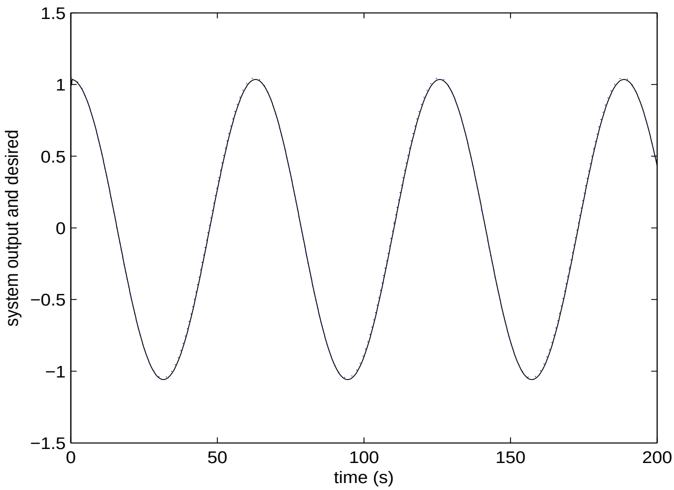


Fig. 3. Responses of $y(t)$ and $y_\gamma(t)$.

3. I. Lagrat, H. Ouakka and I. Boumhidi. "Fuzzy Sliding Mode PI Controller for Nonlinear Systems". WSEAS Transactions on Signal Processing, volum.2:

pp.1137-1143, Sept 2006.

4. Z. Kovacic, M. Balenovic and S. Bogdan. "Sensitivity-based self learning fuzzy logic control for a servosystem". IEEE Control Systems. June 1998.
5. H. Lee and M. Tomizuka. "Robust Adaptive Control using a Universal approximator for SISO Nonlinear Systems". IEEE Transaction on Fuzzy Systems, volum.8: pp.95-106, 2001.
6. N. Golea, A. Golea and K. Benmahammed. "Stable indirect fuzzy adaptive control". Elsevier, Fuzzy Sets and Systems, volum.137, no 3: pp.353-366, 2003.
7. M. Hojati and S. Gazor. "Hybrid Adaptive Fuzzy Identification and Control of Nonlinear Systems". IEEE Transactions on Fuzzy Systems, volum.10, no.2: pp.198-210, 2002.
8. F. Qiao, Q.M. Zhu, A. Winfield and C. Melhuism. "Fuzzy Sliding Mode Control for discrete Nonlinear Systems". Transactions of china automation society, volum.22, no.2, June 2002.
9. L. X. Wang. "Adaptive fuzzy systems and control: designing and stability analysis". Prentice Hall, 1994.
10. J.J. Slotine, "Sliding Controller Design for Non Linear Systems." International Journal of Control, vol.40, no.2, pp. 421-434, 1984.
11. J. Boumhidi and M.Mrabti, "Sliding mode controller for robust force control of hydraulic servo-actuator", In ISEE, IEEE International Symposium on Electrical Engineering, Targoviste, Romania, pp. 27-33, November 1-2, 2004.

This page intentionally left blank

AUTHOR INDEX

Aïssaoui, N., 234
Achkar, Y., 290
Addou, A., 101
Aharouch, L., 170, 181
Akdim, Y., 327, 344
Alaoui Hafidi, Y., 355
Azroul, E., 123, 181

Bahri, A., 1
Bendahmane, M., 205
Benkirane, A., 17, 30, 170
Bennouna, J., 170, 327
Benouna, J., 30
Boucherif, A., 54
Boumhidi, I., 355
Boumhidi, J., 355

Chrif, M., 17, 205

El Habib, S., 278
El Hachimi, A., 72
El Khalil, A., 43
El Manouni, S., 17, 205

Hage Chehade, F., 141

Igbida, N., 266

Lamrani Alaoui, A., 72
Lidouh, A., 101

Marrakchi, A. L., 290
Mekkour, M., 327

Redwane, H., 150
Rhoudaf, M., 30, 123, 181, 327

Sadek, M., 141
Sayouri, S., 290
Seddoug, B., 101

Talbi, M., 217
Touzani, A., 170
Tsouli, N., 217, 278

Youssfi, A., 88